# Declining Dynamism, Increasing Markups and Missing Growth: The Role of the Labor Force*

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**Abstract**

A growing body of empirical research highlights substantial changes in the US economy during the last three decades. Business dynamism is declining, market power seems to be on the rise, and aggregate productivity growth is sluggish. We show analytically that a decline in the rate of growth of the labor force implies all of these features in a frontier model of firm dynamics. The reason is that a decline in labor force growth reduces the long-run entry rate but keeps expansion incentives of existing firms unchanged. This implies that lower labor force growth reduces creative destruction, increases average firm size and raises market power. We calibrate the model to data on employment and sales for the universe of firms in the U.S. Census. We find that the empirically observed decline in the rate of labor growth since the 1980s can account for the decline in entry and the increase in firm size, generates quantitatively significant and welfare-relevant changes in markups, but plays a minimal role in explaining the decline in aggregate productivity growth..

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# 1 Introduction

The U.S. economy has seen substantial changes in the past decade. First, a large body of work has documented a decline in business dynamism. The start-up rate has been on a secular decline, measures of job reallocation have fallen substantially and creative destruction seems to be less potent (Haltiwanger et al., 2015; Pugsley and Şahin, 2015). Second, new evidence is emerging that industry concentration, whether measured by sales or employment, has increased markedly, as have measures of markups (Autor et al., 2017; De Loecker and Eeckhout, 2017). Lastly, save for the I.T.-fueled boom of the late 1990s, productivity growth has been sluggish for many years (Fernald, 2015).

In this paper we argue that standard models of growth and firm dynamics suggest a parsimonious explanation for all these trends: declines in the rate of labor force growth. The main mechanism is quite simple. A long-run stationary firm size distribution implies that the number of firms has to grow at the same rate as the labor force. Hence, a falling rate of labor force growth will translate in a fall in the entry rate of new firms. Because entry is an important component of creative destruction, this decline in the start-up rate ripples through the economy. Incumbent firms face less competition from new firms and find it easier to expand and to raise their prices. As a result, average firm size increases, the pace of job reallocation slows and markups rise. At the same time, the decline in creative destruction also reduces productivity growth at the aggregate level.

To make this intuition precise, we present a new model of firm dynamics that is rich enough to match many first-order features of the process of firm dynamics. As in Garcia-Macia et al. (2016), our model allows for creative destruction (by both entrants and incumbents), the creation of new varieties (again by both entrants and incumbents) and improvements in the productivity of existing firms. In order to introduce a notion of market power, we embed this model in a framework of imperfect product markets, in which markups are determined endogenously. Despite its richness, the model can be solved analytically, and yields tight predictions for the effects of a decline in population growth.

Most importantly, we show that the equilibrium rate of creative destruction by incumbent firms is independent of population growth. Hence, the entirety of the adjustment to a lower rate of population growth falls on entering firms. This directly implies that lower population growth reduces the rate of aggregate creative destruction. We further show that the equilibrium degree of creative destruction is a key determinant of the degree of market power. In particular, as in Peters (2018), creative destruction is pro-competitive because it lowers the extent of markup growth over the firms' life-cycle. Lower population growth therefore increases markups, lowers allocative efficiency and raises corporate profits at the expense of static factors of production like labor.

We then quantify the strength of this mechanism. To do so, we calibrate our model to information on the process of firm dynamics for the population of US firms. Crucially, by linking firm-level information on sales to the U.S. Census, we can measure firm-level markups in a consistent way for all firms in the US, and hence explicitly target the life-cycle profile of markups, which is a key aspect of our theory. With the calibrated model in hand, we ask a simple question: what are the

implications of the observed decline in the rate of labor force growth since 1980? Empirically, the labor force growth almost halved from 2% to 1%. We find that this decline can explain almost the entirety of the decline in the entry rate and the increase in average firm size. We also find that it predicts a quantitatively significant rise in markups across firms.

However, its role in producing slower growth is limited. While creative destruction declines, and the economy sees less growth in varieties, there are powerful offsetting effects on the incentives of incumbents to improve their products. In particular, since incumbents face less competition and each product is less likely to be destroyed, the value of improving their existing products in order to raise markups increases. This leads to a rise in own incumbent innovation that - in our calibration - offsets much of the growth slowdown from lower entry.

Throughout the paper, we take movements in the labor force to be exogenous to market concentration and firm dynamics. In the data, slowing labor force growth reflects three primary channels: an end to increasing female participation, declining prime-age male participation, and slowing growth in the working age population (due to both falls in fertility and slowing immigration). While a declining labor share and rising market power may itself have implications for worker participation, here the simplicity of taking these movements as given yields substantial insight into the changing patterns of firm dynamics we see in the data.

**Related Literature.** We are not the first to connect the decline in the growth rate of the labor force to changes in firm dynamics. Karahan et al. (2016) and Hathaway and Litan (2014) are early contributions that use geographic variation in the age structure of the population of the U.S. to provide direct support that a lower rate of population growth reduces the start-up rate. Recently, Hopenhayn et al. (2018) document the relationship between changes in demographics and firm dynamics in a quantitative model. In contrast, our model highlights that a declining rate of population growth also affects the extent of market power and aggregate productivity growth, and hence has potentially a much broader macroeconomic impact.

On the theory side, we build on firm-based models of Schumpeterian growth in the tradition of Aghion and Howitt (1992) and Klette and Kortum (2004). We augment these model by allowing for quality improvements of existing firms as in Atkeson and Burstein (2010), Akcigit and Kerr (2015) or Luttmer (2007), the creation of new varieties as in Young (1998) and endogenous markups through Bertrand competition as in Peters (2018) or Acemoglu and Akcigit (2012). Our model is thus a version of Garcia-Macia et al. (2016), augmented by endogenous markups and fully endogenous innovation choices, and incorporating long run growth in the labor force.

There is an active literature on the decline in business dynamism in the US. Empirically, various studies show that the entry rate has fallen substantially (Karahan et al., 2015; Alon et al., 2018; Decker et al., 2014), that broad measures of reallocation are lower than in the 80s and 90s (Haltiwanger et al., 2015; Davis and Haltiwanger, 2014), that industries are becoming more concentrated (Kehrig and Vincent, 2017; Autor et al., 2017) and that markups are rising (Edmond et al., 2018; De Loecker and Eeckhout, 2017). See also Akcigit and Ates (2019a) for a summary. In terms of explanations for these

phenomena, Aghion et al. (2019) argue that improvements in IT technology raised the returns to scale and induced firms with high productivity and high markups to expand. Akcigit and Ates (2019b) focus on changes in the process of knowledge diffusion. Our paper is complementary to these studies by highlighting that the secular decline in population growth might be a key factor explaining the low-frequency trends of concentration, markups and growth.

The structure of this paper is as follows. In the next section, we describe the theory and prove our main results on the effects of a decline in the rate of population growth. In Section 3 we calibrate our model to the data on firm dynamics and markups for the population of US firms and quantify the strength of this mechanism. Section 4 concludes. We relegate most proofs of our results to the Appendix.

## 2 Theory: Population Growth and Firm Dynamics

In this section, we present a model of innovating firms in an environment of aggregate labor force growth. We then analyze the link between the rate of growth of the labor force, equilibrium creative destruction and firm entry. We conclude the section by discussing a special case of the model which allows for closed-form solutions and hence provides a tight mapping from the rate of labor force growth to the distribution of markups and equilibrium productivity growth.

### 2.1 General Environment

Time is continuous. There is a measure $L_t$ of identical households who each supply one unit of labour inelastically. This population of households (and hence the labor force) is growing at a constant rate $\eta$. Studying the response of markups, growth and business dynamism to changes in $\eta$ is the main focus of this paper.

Households have preferences over a final consumption good $c_t$, which are given by

$$U = \int_0^\infty e^{-\rho t} \ln(c_t)\, dt.$$

Final consumption is composed of differentiated varieties. As in Klette and Kortum (2004), we model these varieties as particular product lines, which may be produced by multiple firms. Production of the final good takes place in a competitive final sector, with output of the final good given by combining intermediate product lines according to

$$Y_t = \left( \int_0^{N_t} \left( \sum_{f \in S_{it}} y_{fit} \right)^{\frac{\sigma}{\sigma-1}} di \right)^{\frac{\sigma}{\sigma-1}}. \tag{1}$$

Here $N_t$ is the number of active product lines, where these product lines are indexed by $i$. This number evolves endogenously with the creation and destruction of new products. $S_{it}$ is the set of firms actively producing product $i$ at time $t$, and $y_{fit}$ is the production of firm $f$ in product line $i$ at time $t$. Note that this production function implies that the output of firms producing the same product are taken as perfect substitutes in the aggregation into final goods, but that different products are imperfect substitutes, with the elasticity of substitution given by $\sigma$.

Firms can be active in multiple product markets. Each firm $f$ is characterized by a portfolio of the products they produce, denoted by $\Theta_f$ , and the productivity with which they produce each product in their portfolio, indexed by $\{q_{fi}\}_{i\in\Theta_f}$. We denote the number of products firm $f$ produces by $n_f$. Production of each good uses only labour, and is given by

$$y_{fi} = q_{fi} l_{fi}, \tag{2}$$

where $l_{fi}$ is the amount of labour hired by firm $f$ to produce product $i$.

Within product markets, firms compete *a la* Bertrand. Given the CES structure of demand, each firm would like to charge a markup of $\frac{\sigma}{\sigma-1}$ over marginal cost. However, the presence of competing firms implies that the most efficient producer of the good might have to resort to limit pricing. If they are unable to price at the optimal markup without inviting competition, they will set their price equal to the marginal cost of the next most efficient producer of that good, who is then indifferent between producing or not. Given the production function in (2), the markup charged in product $i$, $\mu_i$, is given by

$$\mu_i = \min\left\{\frac{\sigma}{\sigma-1}, \frac{q_i}{q_i^C}\right\}, \tag{3}$$

where $q_i$ denotes the productivity of current producer in product $i$ and $q_i^C$ is the productivity of the most efficient competitor. Equation (3) highlights that markups are rising in a firm's productivity advantage relative to their competitors. Hence, the distribution of markups is jointly determined with the distribution of relative productivities in equilibrium.

It is easy verify that product-level sales $py_i$, employment $l_i$ and profits $\pi_i$ are given by

$$py_i = \mu_i^{1-\sigma}\left(\frac{q_i}{w_t}\right)^{\sigma-1} Y_t, \qquad l_i = \mu_i^{-\sigma}\left(\frac{q_i}{w_t}\right)^{\sigma-1}\frac{Y}{w}, \qquad \pi_i = \left(1-\frac{1}{\mu_i}\right)\mu_i^{1-\sigma}\left(\frac{q_i}{w}\right)^{\sigma-1} Y. \tag{4}$$

For the case of $\mu = \frac{\sigma}{\sigma-1}$, these expressions reduce to the usual expressions where profits and labor payments are a constant share of total revenue, i.e. $\pi_i = \frac{1}{\sigma} py_i$ and $wl_i = \frac{\sigma-1}{\sigma} py_i$. Because firms produce multiple products, firm-level sales are $py_f = \sum_{i=1}^{n_f} py_i$ (and for profits and employment analogously).

Given the firm-level allocations in (4), we can also characterize the aggregate allocations. In particular

aggregate output $Y_t$ and the level of wages $w_t$ are given by

$$Y_t = \mathcal{M}_t Q_t N_t^{\frac{1}{\sigma-1}} L_t^P \text{ and } w_t = \Lambda_t Y_t / L_t^P$$

where $Q_t = \left(\int q_i^{\sigma-1} dF_t(q)\right)^{\frac{1}{\sigma-1}} = \left(E_t\left[q_i^{\sigma-1}\right]\right)^{\frac{1}{\sigma-1}}$ and

$$\mathcal{M}_t = \frac{\left(\int_i \mu_i^{1-\sigma} (q_i/Q_t)^{\sigma-1} di\right)^{\frac{\sigma}{\sigma-1}}}{\int_i \mu_i^{-\sigma} (q_i/Q_t)^{\sigma-1} di} \quad \text{and} \quad \Lambda_t = \frac{\int_i \mu_i^{-\sigma} di}{\int_i \mu_i^{1-\sigma} di}. \tag{5}$$

The two aggregate statistics $\mathcal{M}_t$ and $\Lambda_t$ fully summarize the static macroeconomics consequences of monopoly power. Market power reduces both production efficiency (the misallocation term $\mathcal{M}_t$) and lowers factor prices relative to their social marginal product (the labor share $\Lambda_t$). In particular, a common increase in markups reduces the labor wedge $\Lambda_t$ but keeps the allocation efficiency $\mathcal{M}_t$ unchanged. The latter is affected by the dispersion of markups. Because our model generates the distribution of markups endogenously and this distribution is a function of the rate of population growth, changes in the rate of population growth directly affect $\mathcal{M}_t$ and $\Lambda_t$.

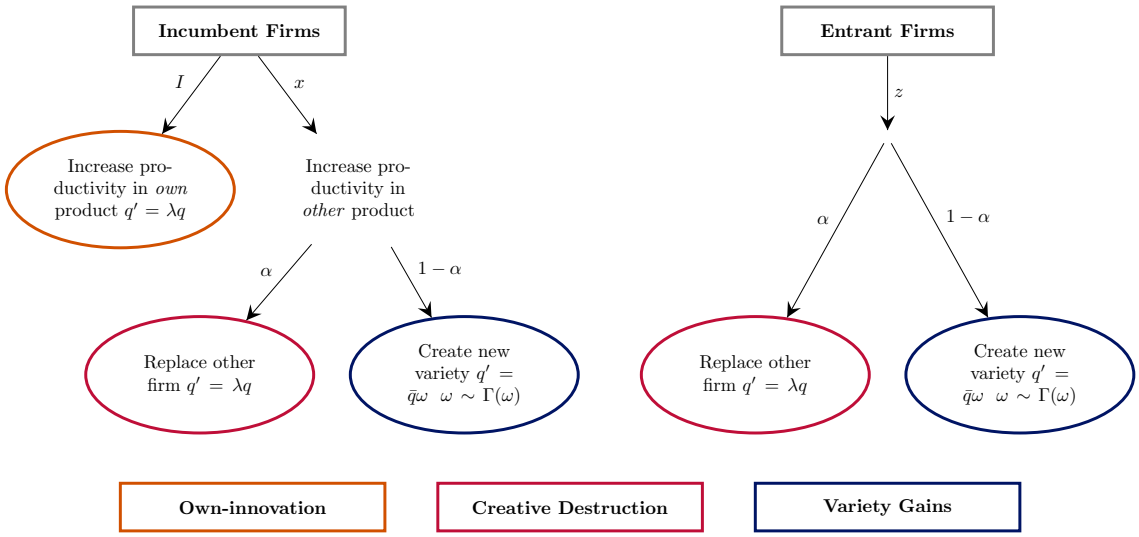
**Firm Dynamics and Aggregate Growth.** Both firms' productivities and the products they sell evolve endogenously. As in Garcia-Macia et al. (2016), our theory allows for three sources of firm dynamics. First we allow for *creative destruction* by incumbents and entrants (as in Klette and Kortum (2004)). Creative destruction occurs when either an existing firm or a new firm improves the productivity of a product $i$, which is currently produced by another producer. Because the output of firms producing the same product $i$ is considered to be perfectly substitutable (see (1)), such productivity improvements result in churning, whereby the old producer gets replaced.

Second we allow for *own-innovation*, whereby firms can improve the productivity $q$ of the products they are currently producing (see Atkeson and Burstein (2010) or Luttmer (2007)).

Third, we allow for the endogenous entry of *new varieties*. This margin is the source of variety gains, whereby firms can generate product varieties which are entirely new to the economy. As in Young (1998), it is this margin which implies that the model does not suffer from strong scale effects, i.e. the growth rate, while still endogenous, is independent of the level of the population.

We formalize these decisions in the following way (see Figure 1 for a graphical depiction). We assume that productivity $q$ evolves on a quality ladder and that each successful innovation represents a productivity improvement of $\lambda > 1$. We also refer to $\lambda$ as the step size. Existing firms can spend resources to engage in own-innovation and improve the productivity of the products in their existing portfolio. In particular they can choose a Poisson rate $I_i$ at cost $c_t^I(I_i, q_i)$ (denominated in units of labor) to improve the productivity for product $i$ from $q_{fi}$ to $\lambda q_{fi}$. They can also try to expand into new products by choosing the Poisson rate $X$ (at a cost $c_t^X(X, n)$, where $n$ denotes the number of products the firm is currently producing and $X$ denotes the flow rate of innovation) to improve the productivity of product new to them. Conditional on successfully creating a new product, this prod-

Figure 1: Dynamic Decisions: Entry and Innovation




Note: This figure describes the innovation choices available to both incumbent and entrant firms in the model.

uct can either be a new variety or it can improve upon an already existing product from another firm. We assume that innovation is undirected, such that the firm cannot target new or existing varieties. With probability $\alpha$ the product represents a technological advance over an existing firm, increases the product's productivity by $\lambda$ and forces the current producer to exit ("incumbent creative destruction"). With the complementary productivity $1-\alpha$, the product will be new to the society as a whole. We assume that the quality of new products is given by $q^{\sigma-1} = \omega\overline{q}_t$, where $\omega$ is drawn from a fixed distribution $\Gamma(\omega)$ and $\overline{q}_t = E_t\left[q_i^{\sigma-1}\right]$. Hence, as in Buera and Oberfield (2016), the productivity of new varieties is determined both by the existing knowledge embedded in $\bar{q}_t$ and by novel ideas.

Entrants have the same opportunities as incumbent firms. While they naturally do not own any products they could improve on, entering firms also engage in creative destruction and new variety creation. As for incumbent firms, the share of innovations, which result in creative destruction is exogenous and given by $\alpha$. Entrants have access to a linear entry technology, where each worker generates a flow of $\vartheta_E$ ideas.

Finally, we also assume that any product can die exogenously at a constant Poisson rate $\delta$. If a product line dies, the knowledge of how to produce it is lost forever, to all those who knew how to produce it. Allowing for this possibility is not conceptually important but it gives us one additional degree of freedom for our quantitative assessment.

**Endogenous Market Power.** A key aspect of our theory is that the distribution of markups is endogenous and jointly determined with the process of firm dynamics outlined above. The assumption of the quality ladder structure implies that the equilibrium markup in (3) is given by $\mu_i = \min\left\{\frac{\sigma}{\sigma-1}, \lambda^{\Delta_i}\right\}$, where $\Delta_i$ is the number of quality steps the current producer is ahead of the second most productive firm. Hence, the equilibrium markup is fully determined from the produc-

tivity gap $\Delta_i$. In particular, as long as $\Delta_i < \overline{\Delta} \equiv \frac{\ln(\sigma/\sigma-1)}{\ln\lambda}$, equilibrium prices are determined from limit pricing and markups are increasing in $\Delta_i$.

To see how the endogenous distribution of markups is determined, note the difference between successful creative destruction and own-innovation. Suppose the current producer of product $i$ has a quality gap of $\Delta_i$. If this firm is replaced by another producer through successful creative destruction, the quality gaps *reduces* to unity as the new firm is only a single step ahead of the erstwhile producers. Hence, creative destruction reduces markups. In contrast, if the existing firm successfully increases its productivity through own-innovation, the quality gap *increases* to $\Delta_i + 1$ and so do markups (as long as $\Delta < \overline{\Delta}$). Formally, our model of growth gives rise to an endogenous stochastic process of markups. Let $\tau$ denote the rate of creative destruction and let $I_i$ denote the rate of own-innovation. Within a small time interval $\iota$, the evolution of the markup in product $i$ is given by

$$\mu_{i,t+\iota} = \begin{cases} \lambda & \text{with probabilty } \tau\iota \\ \min\left\{\lambda\mu_{i,t}, \frac{\sigma}{\sigma-1}\right\} & \text{with probabilty } I_i\iota \\ \mu_{it} & \text{with probabilty } 1-(\tau+I_i)\,\iota \end{cases} . \tag{6}$$

Hence, own-innovation is akin to a positive drift for the evolution of markups, while creative destruction is similar to a "reset" shock, which lowers markups: creative destruction is pro-competitive by keeping the accumulation of market power in check. In contrast, firms' own-innovation efforts are the main force driving markups upwards. As we will show below, the stochastic process in (6) gives rise to a stationary distribution of markups along the balanced growth path, which explicitly depends on the endogenous rate of creative destruction and own-innovation. And because both these equilibrium outcomes respond to the rate of population growth $\eta$, a declining rate of population growth will directly affect the equilibrium degree of market power. In fact, we will show that lower population growth increases markups by lowering creative destruction and raising own-innovation efforts.

**Optimal Innovation, Creative Destruction and Entry.** Firms' innovation and expansion decisions are forward-looking. Hence, we need to solve for the value function to characterize optimal behavior. Given the expression for profits in (4) and for markups in (3), the state variables at the firm-level are $[\Delta_i, q_i]_{i=1}^n$. We denote the rate of creative destruction per product as $\tau_t$. This rate is determined endogenously but taken as given by existing firms. Consequently, the value function of a firm is

given by

$$
\begin{aligned}
r_t V_t\left([\Delta_i, q_i]\right) \quad = \quad & \underbrace{\sum_{i=1}^{n} \pi_t\left([\Delta_i, q_i]\right)}_{\text{Profits}} + \underbrace{\dot{V}_t\left([\Delta_i, q_i]\right)}_{\text{Capital Gains}} \\
& + \sum_{i=1}^{n} \underbrace{(\tau_t + \delta_0)\left[V_t\left(\left[\Delta_j, q_j\right]_{j \neq i}\right) - V\left([\Delta_i, q_i]\right)\right]}_{\text{Creative Destruction and Product Loss}} \\
& + \max_{I} \sum_{i=1}^{n} \left\{ I_i \left[ \underbrace{V_t\left(\left[\Delta_j, q_j\right]_{j \neq i}\left[\Delta_i + 1, \lambda q_i\right]\right)}_{\text{Value of own innovation}} - V_t\left([\Delta_i, q_i]\right) \right] - c^I\left(I_i, q_i\right) w_t \right\} \\
& + \max_{X} \Big\{ X \Big[ \underbrace{\alpha \int V_t\left([\Delta_i, q_i], 1, \lambda q\right) dF_t}_{\text{Replacing an existing firm}} + \underbrace{(1-\alpha) \int V_t\left([\Delta_i, q_i], \overline{\Delta}, \omega \bar{q}_t\right)}_{\text{New variety}} d\Gamma - V_t\left([\Delta_i, q_i]\right) \Big] \\
& - c^X\left(X, n\right) w_t \Big\}.
\end{aligned}
$$

The value of a firms consists of multiple parts. First of all, the value of the firm is increasing in the current flow profits and the capital gain $\dot{V}$. Secondly, the firm might lose any of its products either to another firm (which happens at the endogenous rate of creative destruction $\tau$) or for exogenous reasons. Third, the possibility of own-innovation carries an option value. In particular, a successful event of own-innovation in product $i$ raises the quality $q_i$ by $\lambda$. By increasing the quality gap it also increases markups (provided that $\Delta_i + 1 < \overline{\Delta}$). Finally, the firm has the option to improve the productivity of a product outside its portfolio. With probability $\alpha$ it replaces a randomly selected product in which case the quality gap gets "reset" to unity. With probability $\alpha$, the firms creates a new variety, whose quality is given by $\omega \bar{q}$, where $\omega \in \Gamma$ and the markup is given by $\sigma / (\sigma - 1)$.

The free entry condition requires that the expected value of a successfully created new product (which, with probability $\alpha$, stems from an existing firm and with probability $1 - \alpha$ is entirely new to society) does not exceed the cost of entry, i.e.

$$
\alpha \int_q V_t\left(1, \lambda q\right) dF_t\left(q\right) + (1-\alpha) \int_q V_t\left(\overline{\Delta}, \omega \bar{q}_t\right) d\Gamma\left(\omega\right) \leq \frac{1}{\vartheta_E} w_t. \tag{7}
$$

For the remainder of this paper we focus on allocations where the flow of entry is positive and (7) holds with equality.

**Equilibrium and Balanced Growth.** To close the model in general equilibrium, labor market clearing requires that the total population $L_t$ equals the number of production workers $L_{Pt}$ and the number of workers engaged in own-innovation, incumbent expansion and entry, i.e.

$$
L_t = L_{Pt} + \frac{1}{\varphi_z} Z_t + \int_i c^I\left(I_i, q_i\right) di + \int_i c^X\left(x_i, 1\right) di.
$$

Consistency also requires that the aggregate rate of creative destruction is consistent with firms'

expansion and entry incentives. Formally,

$$\tau_t = \alpha \frac{\int_i x_i di + Z_t}{N_t}. \tag{8}$$

Recall that the mass of active products is given by $N_t$ and that we defined $\tau_t$ as the rate of creative destruction per product.

In this paper we focus on stationary equilibria, where the distributions of firm size and markups are constant and the economy-wide aggregates grow at a constant rate. These equilibria are defined as follows:

**Definition.** *A stationary equilibrium is a set of allocations* $\{l_{it}, I_{it}, x_{it}, z_t, y_{it}, c_t\}_{it}$ *and prices* $\{w_t, r_t, p_{it}\}_{it}$ *such that*

(i) *All aggregate variables grow at a constant rate*

(ii) *Consumers chose* $[y_{it}, c_t]_{it}$ *to maximize utility*

(iii) *Firms chose* $[I_{it}, x_{it}, p_{it}]$ *optimally*

(iv) *The free entry condition is satisfied*

(v) *All markets clear*

(vi) *The cross-sectional distributions of markups and firm size are stationary.*

## 2.2 The Main Result: Population Growth, Entry and Creative Destruction

In this section we derive the central result of this paper. We show that as long as the economy is on a balanced growth path (BGP), lower population growth reduces the rate of entry and the rate of creative destruction but keeps incumbents' incentives to expand unchanged.

Consider a BGP where the firm size distribution is stationary and all aggregate variables grow at a constant rate. This implies that the number of products $N_t$ has to grow at the same rate as the labor force, i.e.

$$\eta = \frac{\dot{N}_t}{N_t}. \tag{9}$$

Furthermore, incumbents' expansion rate *per product* $x$ and the entry flow *per product* $z$ are constant. Equation (8) then implies that that the rate of creative destruction per product $\tau$ is given by

$$\tau = \alpha (x + z) \tag{10}$$

as a share $\alpha$ of entry and expansion activities result in a replacement of existing firms. Similarly, the evolution of the of the number of products $N_t$ is given by

$$\dot{N}_t = (1 - \alpha)(x + z) N_t - \delta N_t. \tag{11}$$

These three equations directly imply that

$$\tau = \frac{\alpha}{1-\alpha}(\eta + \delta),$$

i.e. the rate of aggregate creative destruction $\tau$ is increasing in the rate of population growth.

Because creative destruction is due to both entrants and incumbents (see (8)), the declining rate of population growth could reduce creative destruction either because it reduces entry or because it reduces creative destruction by incumbents (or both). We now show that it is in fact *only* entrants which are affected. To see this, note that (as we show in detail in Section A.2 in the Appendix) the value function defined above is additive across products, i.e.

$$V_t([\Delta_i, q_i]) = \sum_i V_t(\Delta_i, q_i).$$

This directly implies that the optimal rate of incumbent product innovation is given by

$$x = \arg\max_x \left\{ x \left[ \alpha \int V_t(1, \lambda q)\, dF_t + (1-\alpha) \int V_t(\overline{\Delta}, \omega \bar{q}_t)\, d\Gamma \right] - \frac{1}{\vartheta_x} x^{\zeta} w_t \right\},$$

as $V_t([\Delta_i, q_i], 1, \lambda q) - V_t([\Delta_i, q_i]) = V_t(1, \lambda q)$. The free entry condition in (7) therefore directly implies that

$$x = \arg\max_x \left\{ x \frac{1}{\vartheta_E} w_t - \frac{1}{\vartheta_x} x^{\zeta} w_t \right\} = \left( \frac{1}{\zeta} \frac{\vartheta_x}{\vartheta_E} \right)^{\frac{1}{\zeta - 1}},$$

i.e. the equilibrium rate of product creation by incumbents is constant and only depends in technological parameters. Crucially, it is independent of the rate of population growth $\eta$.

Because these two results are central for the economic mechanism of this paper, we gather them in the following Proposition.

**Proposition 1.** *Consider a BGP. The rate of creative destruction $\tau$, the rate of incumbent product creation $x$ and the rate of entry $z$ is given by*

$$\tau = \frac{\alpha}{1-\alpha}(\eta + \delta) \qquad x = \left( \frac{1}{\zeta} \frac{\vartheta_x}{\vartheta_E} \right)^{\frac{1}{\zeta - 1}} \qquad z = \frac{\eta + \delta}{1-\alpha} - x. \tag{12}$$

*Hence, population growth increases creative destruction, $\frac{\partial \tau}{\partial \eta} > 0$, increases the flow rate of entry, $\frac{\partial z}{\partial \eta} > 0$, but leaves incumbent expansion unchanged $\frac{\partial x}{\partial \eta} = 0$.*

*Proof.* See Section A.2 in the Appendix. □

Proposition 1 contains two key theoretical results of this paper. First of all, a decline in population growth reduces creative destruction. Secondly, the entirety of the decline is absorbed by the

economy's extensive margin - entrants do all the work. Hence, even though our model allows for incumbents' incentives to engage in product creation to change, in equilibrium free entry implies that incumbents' rate of product creation is insulated from demographics.

Two assumptions are driving this result. First of all, our assumption that entry and incumbent expansion is undirected implies that creative destruction is tied to the creation of new products. Below we argue that this assumption is easy to dispense with - as long as there are some complementarities that tie the incentives to create new varieties to the incentives to create better versions of existing products which are new to the firm, the rate of population growth will affect the rate of creative destruction in equilibrium. Secondly, we assume that entrants' technology to create new products is linear, while incumbents' technology is convex (at the firm-level). While we find this assumption very natural, we also note that most models of firm dynamics rely on this assumption. Hence, to the extent that these models are credible to analyze the process of firm dynamics, they have strong implications for the effects of a decline in population growth.

Armed with Proposition 1 we now trace out its implications. In particular, we show that lower population growth increases market power, concentration and firm size and lowers the equilibrium entry rate and aggregate productivity growth.

### 2.3 Implications: Population Growth, Dynamism, Markups and Growth

To trace the implications of a reduction in population growth, we start our analysis with a special case of the general framework above. We impose two restrictions. First of all, we assume that consumer preferences take the Cobb Douglas form, i.e. $\sigma \to 1$. Secondly, we assume the incumbent own-innovation $I$ is exogenous, such that existing firms improve the quality of their own products at a constant rate.

This case is analytically attractive because we can solve the entire model in closed form. For our quantitative analysis in Section 3 we are of course not imposing these restrictions. The key simplifications of these assumptions are threefold. First of all, the limit price is always binding, i.e. the markup in product $i$ is given by $\mu_i = q_{i/}/q_i^C = \lambda^{\Delta_i}$.[1] Secondly, firm sales are not a function of productivity $q$ but only of the number of products $n$ - see (4). This implies that we can tightly characterize the distribution of sales and its dependence on $\eta$. Finally, our assumption that $I$ is exogenous allows us to fully characterize both the distribution of markups and the equilibrium growth rate in terms of parameters, including labor force growth $\eta$.

#### Result 1: Population Growth and Market Power

We first turn to the effect of population growth on the equilibrium degree of market power.

[1] Because markups are only constrained by the limit price, we assume that newly invented products charge a markup $\lambda$, i.e. as if they had a productivity advantage of a single step.

**Proposition 2.** *Consider a BGP. Define*

$$\gamma(\eta) = \ln\left(1 + \frac{\eta + \delta_0}{(1-\alpha) I}\right)$$

*Then:*

*(i) The stationary distribution of markups is given by*

$$G(\mu) = 1 - \mu^{-\frac{1}{\ln\lambda}\gamma(\eta)}$$

*(ii) Aggregate misallocation is given by*

$$\mathcal{M} = e^{-\frac{1}{\gamma(\eta)}} \frac{1+\gamma(\eta)}{\gamma(\eta)} \text{ and } \Lambda = \frac{\gamma(\eta)}{1+\gamma(\eta)}.$$

*Hence, lower population growth increases markups and misallocation and reduces the labor share.*

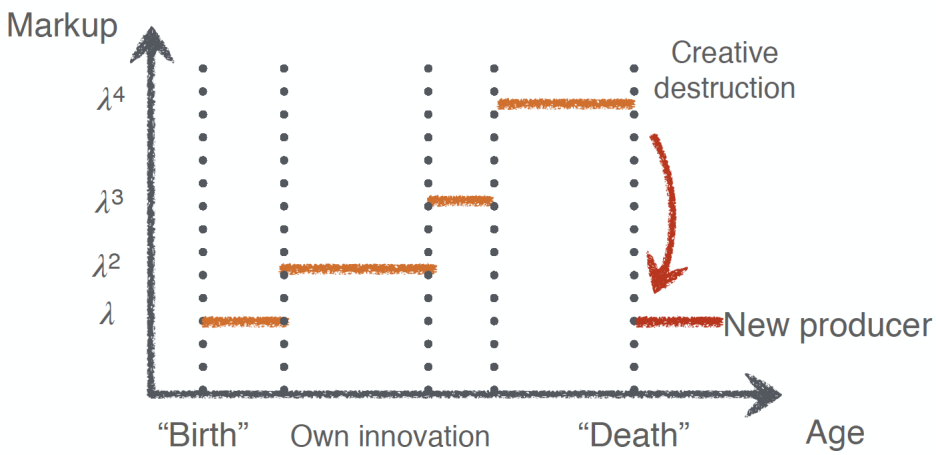
*Proof.* See Section A.4 in the Appendix. □

Proposition 2 concisely links the rate of population growth to the stationary degree of market power. Our model implies that the unique equilibrium distribution of markups is a Pareto distribution, whose shape parameter $\gamma$ is increasing in the rate of population growth. Hence, lower population growth *increases* both the level and the dispersion of markups.

This result is similar to the finding in Peters (2018), who studies a related model without population growth and without the new variety margin. He shows that the Pareto tail of the markup distribution is determined by the rate of creative destruction $\tau$. Proposition 2 leverages this result and the fact that creative destruction is directly tied the rate of population growth. The intuition is simple and displayed in Figure 2. Conditional on survival, markups increase stochastically at rate $I$. Conversely, creative destruction $\tau$ resets the quality gap within a product and hence reduces markups. Creative destruction therefore lowers the expected time a given firm produces a particular product. Hence, as in Jones and Kim (2016), creative destruction plays the role of a "death shock", which limits the extent to which incumbent firms can accumulate market power through successful draws of own-innovation. The resulting stationary distribution of the stochastic process in (6) is the Pareto distribution of Proposition 2.

The second part of the proposition contains expressions for the macroeconomic consequences of misallocation. In the case of $\sigma = 1$, the two statistics $\mathcal{M}$ and $\Lambda$ only depend on the marginal distribution of markups (see (5)) and can be solved in closed form. It is easy to verify that they are increasing in $\gamma(\eta)$. Lower population growth increases misallocation through lower creative destruction and more dispersed markups. Lower population growth also lowers the production workers' labor share $\Lambda$ at the expense of higher profits.

Figure 2: The Life-Cycle of Product Markups




Note: This figure shows a stylized example of how markups evolve at the product level: from when a firm takes over a product, through a sequence of own innovations in which mark-ups increase, to a creative destruction event by another firm which reduces the markup to the baseline level.

**Result 2: Population Growth and Aggregate Growth**

As usual in models of Schumpeterian growth, in our economy creative destruction is an important determinant of aggregate productivity growth. This already suggests an important channel why lower population growth might indeed be detrimental for the rate of productivity growth. This intuition is indeed correct.

**Proposition 3.** *Consider a BGP. The growth rate of GDP per capita $g$ is given by*

$$g = \ln(\lambda)(\tau + I) = \ln(\lambda)\left(\frac{\alpha}{1-\alpha}(\eta + \delta_0) + I\right) \tag{13}$$

*Hence, lower population growth reduces productivity growth.*

*Proof.* See Section A.3 in the Appendix. □

Proposition 3 shows why population growth is a driver of income growth: it affects creative destruction, which itself is an important component of aggregate productivity growth. Proposition 3 also highlights additional effects, which are absent in this special case of our theory but present in the full model. By assuming that $I$ is exogenous, we abstract from any feedback between creative destruction and incumbent own-innovation. In our full model, $I$ will also be endogenous and hence respond to changes in the rate of population growth. Furthermore, in the Cobb-Douglas case of the model, there are no variety gains, i.e. growth in the number of products does not affect the growth rate of income

per capita. Again: variety will be present in our full model. We will show below that the relationship between $g$ and $\eta$ is negative even once these additional aspects are taken into account.

**Result 3: Population Growth and Firm Dynamics**

Finally, we turn to the implications for the process of firm dynamics. We are particularly interested in the the effect of population growth on the start-up rate and average firm size. That lower population growth indeed reduces the start-up rate and increases firm size, follows almost immediately from Proposition 1: lower population is entirely absorbed by the entry of new firms. This naturally reduces the entry rate. At the same time, incumbent firms face less competition from new entrants and hence have an easier time to expand. In a stationary equilibrium this raises average size.

**Proposition 4.** *A reduction in population growth lowers the entry rate and increases average firm size and concentration.*

# 3 Quantifying the Effects of Lower Labor Force Growth

The theoretical results above highlighted the importance of the rate of population growth for creative destruction, concentration markups and growth. In this section we calibrate our model to quantify these effects. We consider the full model, with endogenous innovation choices by incumbents, and CES preferences over varieties, such that $\sigma > 1$.

Our exercise is conceptually simple. We calibrate the model to a balanced growth path matching key moments of the data from 1980, when labor force growth was 1.8%. We then reduce the rate of labor force growth $\eta$ by 1%, the magnitude of the decline observed until 2008, and trace out the implications for the entry rate, average firm size, markups and lifecycle growth in the sequence of BGP's parameterized by $\eta$.

## 3.1 Data

We use data from the U.S. Census Longitudinal Business Database (LBD) to calibrate the lifecycle growth of sales and markups by age. The LBD is an administrative dataset containing information on the universe of employer establishments since 1978. It contains information on the age, headcount, industry, employment and payroll of each establishment, along with identifiers at the firms level that allows us to track the ownership of each establishment over time.

To measure firms' markups, we obviously require information on sales. Hence, we augment the LBD data with information on firm revenue from administrative data contained in the Census' Business Register, following Moreira (2015) and Haltiwanger et al. (2016). The Business Register is the master list of establishments and firms for the U.S. Census and we are able to match approximately 70% of the records to the LBD.

We measure markups at the firm level by the inverse labor share, i.e. the markup of firm $f$ is given by

$$\mu_f = \frac{py_f}{wl_f}, \tag{14}$$

where $py_a$ is the total revenue of the firm, and $wl_a$ is the total wage bill. We calculate the total wage bill by aggregating establishment payroll. In terms of our theory, (4) implies that this average markups is given by $\mu_f = \sum_i \mu_i \frac{l_i}{\sum_i^{N_f} l_i}$,i.e. is an average of the product-level markup $\mu_i$ weighted by the employment (or cost) shares (see also Edmond et al. (2018)). While (14) allows us in principle to measure markups for the population of U.S. firms, we only use the life-cycle of markups to estimate our model. More specifically, letting $\mu_f(a)$ be the mark-up of firm $f$ at age $a$, we measure the life-cycle markup growth of firm $f$ until age $a$ as

$$g_f^{\mu}(a) = \frac{\mu_f(a)}{\mu_f(0)}. \tag{15}$$

We focus on (15) for empirical reasons. In particular, the expression in (15) is not specific to our model but holds in more general contexts if one assumes that the output elasticity of the firm with respect to labor is constant with age (see De Loecker and Warzynski (2012) and Peters (2018)). In contrast, if firms within sectors had different production function with different output elasticities, neither the level not the dispersion of markups as measured from (14) could be distinguished from such differences in technology. By focusing on (15) we essentially control for a firm fixed effect when measuring properties of firms' markups.[2]

Lastly, we define the age of the firm in the LBD as the age of the oldest establishment that the firm owns. The birth of a new firm requires both a new firm ID in the Census and a new establishment record. We also modify the Census firm ID's to deal with some issues involving multi-establishment firms in the same way as developed in Walsh (2019).

### 3.2 Calibration

The model is parsimoniously parametrized and only contains 10 parameters

$$\Psi = \left\{ \underbrace{\rho, \sigma}_{\text{Preferences}}, \underbrace{\alpha, \zeta, \vartheta_x, \vartheta_z, \vartheta_I}_{\text{Innovation \& Entry technology}}, \underbrace{\delta_0}_{\text{Exog. exit}}, \underbrace{\eta}_{\text{Pop. growth}}, \underbrace{\lambda}_{\text{Stepsize}} \right\}.$$

Three of them - the discount rate $\rho$, the demand elasticity $\sigma$ and the convexity of the innovation cost function $\zeta$ - we set exogenously. The rate of labor force growth $\eta$ is directly observed in the data. The remaining six parameters are calibrated internally. We target key moments from the cross-sectional

[2]This allows us to avoid estimating output elasticities for labor, which is not feasible with the data we have as it does not contain data on capital of material inputs. Doing so would also complicate the mapping from model to data, since in our model labor is the only factor of production.

Table 1: Structural Parameters

| | Structural Parameters | | Moments | | |
|---|---|---|---|---|---|
| | Description | Value | | Data | Model |
| $\eta$ | Av. labor force growth in 1980-5 | 0.015 | Data from BLS | 1.5 % | 1.5 % |
| $\lambda$ | Step size on quality ladder | 1.004 | Aggregate growth | 2% | 2% |
| $\vartheta_I$ | Cost of inc. own-innovation | 0.18 | Markup growth by age 10 (RevLBD) | 10.2% | 10.2% |
| $\vartheta_X$ | Cost of inc. product creation | 0.25 | Sales growth by age 10 (RevLBD) | 58% | 58% |
| $\vartheta_E$ | Cost of entry | 203 | Avg. firm size (BDS) | 20.5 | 20.5 |
| $\delta$ | Destruction rate of products | 0.048 | Entry Rate in 1980 (BDS) | 14 % | 14 % |
| $\alpha$ | Share of creative destruction | 0.71 | Markup of entrants | 10% | 10% |
| $\zeta$ | Curvature of innovation cost | 2 | Set exogenously | | |
| $\sigma$ | Demand elasticity | 4 | Set exogenously | | |
| $\rho$ | Discount rate | 0.05 | Set exogenously | | |

Note: This table reports the calibrated parameters for the full model with CES preferences over varieties. Data for the firm lifecycle comes from the U.S. Census Longitudinal Database, augmented with revenues from tax-information using the Census' Business Register. Data for average firm size and the firm entry rate in 1980 are taken from the public use Business Dynamics Statistics.

firm-size distribution in 1980 and observed life-cycle dynamics of markups and firm size. While all moments are targeted simultaneously there is nevertheless a tight mapping between particular moments and particular parameters. In Table 1 we report the parameters and the main moments we target.

Because markup growth is driven by incumbents' incentives for own-innovation, we discipline the cost of own-innovation, $\vartheta_I$ from the observed markup profile. Empirically, markups increase on average by ten percentage points for the 10 years of a firm's life-cycle (see also Figure 3 below, where we depict the whole profile). In contrast, sales growth is also affected by the rate of incumbent product creation $x$. Hence, we calibrate $\vartheta_x$ to match average sales growth, i.e. the fact that 10 year old firms are roughly 60% larger than entrants.

We match average firms size and the entry rate by choosing the entry costs $\vartheta_E$ and the rate of product destruction $\delta_0$. Intuitively, the higher $\delta_0$ the lower average firm size as firms keep losing products at a higher rate. Similarly, a higher entry cost $\vartheta_E$ increases incumbent expansion $x$ and reduces the flow of entry $z$ (see Proposition 1). This increases average firm size and reduces the entry rate. Together, these two moments identify $\delta_0$ and $\vartheta_E$. Finally, we pick the step size $\lambda$ to match a given aggregate growth rate of 2% - see 13.

The share of new products in innovation, $1-\alpha$, plays a central role in the level of markups in the economy. In particular, since a new entrant either beings life with a single new product (in which case the markup is $\frac{\sigma}{\sigma-1}$), or steals a single product with a productivity gap of unity (and hence a

markup of $\lambda$), the average markup of entering firms is given by

$$E\left[\mu_f(0)\right] = \alpha\lambda + (1-\alpha)\frac{\sigma}{\sigma-1}.$$

Given $\lambda$ and $\sigma$ we can therefore directly infer $\alpha$ from this moment. We set the average markup of entering firms to 10%.

As shown in Table 1 the model is able to match these moments perfectly. Before using this framework to quantify the effects of declining population growth on the process of firm-dynamics and the macroeconomic aggregates, we will now show that the model is also consistent with some moments, which we did not explicitly target.

### 3.3 Non-Targeted Moments

In Figure 3 we show the model's performance against our targets of sales and markups growth by age. To estimate the lifecycle of relative sales in the data, we regress total firm revenue $py_f$ on a full set of non-parametric age dummies, according to

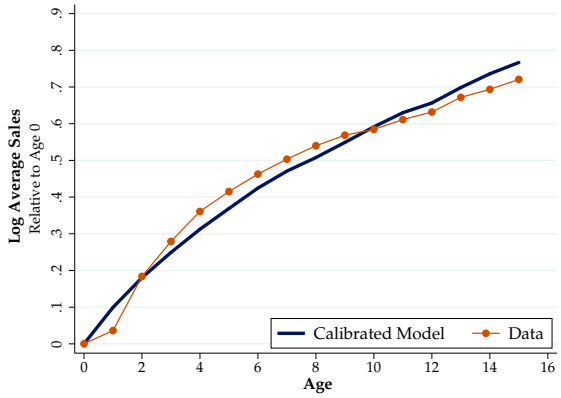
$$\ln py_{f,t} = \sum_{a=0}^{A} \gamma_a^{py} \mathbb{I}_{Age_{ft}=a} + \theta_f + \theta_t + \epsilon_{f,t} \tag{16}$$

where $py_{f,t}$ is the revenue of firm $f$ at time $t$, $\mathbb{I}_{Age_{ft}=a}$ is an indicator for whether the firm is of age $a$and $A$ is a maximum age which we set at 20 and $\theta_f$ and $\theta_t$ are firm and time fixed effects respectively. Hence, $\gamma_a^{py}$ represents the average growth of sales by age $a$. These objects are only observed if a firm survives until age $a$, and we pool estimation for all firms for which we can observe revenue in the LBD. In the model, we simulate a panel of $10^6$ firms using a discrete time approximation to the continuous time dynamics outlined above, and take annual snapshots of the panel of firm sales. Since (by assumption) there is no unobserved heterogeneity for firms in the model, we simply take the age means of revenue within each age group to compare this to the empirical regression. Analogously, we also estimate (16) using the log markup $\ln \mu_{f,t}$ as a dependent variable. In that case, the coefficients $\gamma_a^{\mu}$ represent exactly the average rate of markup growth, i.e. (15). As highlighted in Table 1, we calibrate our model to match $\gamma_{10}^{py}$ and $\gamma_{10}^{\mu}$, i.e. average firm size and the average markup of firms of age 10 relative to entrants.

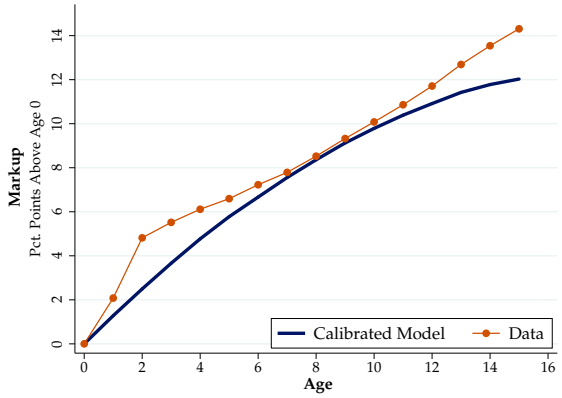
In Figure 3 we display the lifecycle of sales (left panel) and markups (right panel) both in the data and in the model. Even though the model is only calibrated to match the growth from birth to age 10, Figure 3 shows that the whole age profile of sales and markups is quite close in the model to what is observed in the data. In particular for the case of sales, the model replicates the slight concavity of log sales well. In the model, this shape reflects survivorship bias; small firms either grow or are destroyed, while large firms can have products stolen and shrink without exiting. As such, average growth conditional on survival is declining with age for young firms. Quantitatively, firms in the US grow their sales by about 0.6 log points during their first 10 years. The fit for markups in Panel

Figure 3: Lifecycle Growth in Firm Sales and Markups

(a) Sales Growth By Age




(b) Markup Growth By Age




Note: Panel (a) in this Figure compares the lifecycle of firm sales in the model to the estimated lifecycle in the data. The data lifecycle plots the age coefficients from estimating equation (16) in the LBD. $N = \{35,300,000\}$, where this number has been rounded to accord with Census Bureau disclosure rules. The lifecycle of sales in the calibrated model is computed by simulating a panel of $10^6$ firms , and averaging sales within age groups. Panel (b) does the same for relative markups, where these are defined as in equation (15).

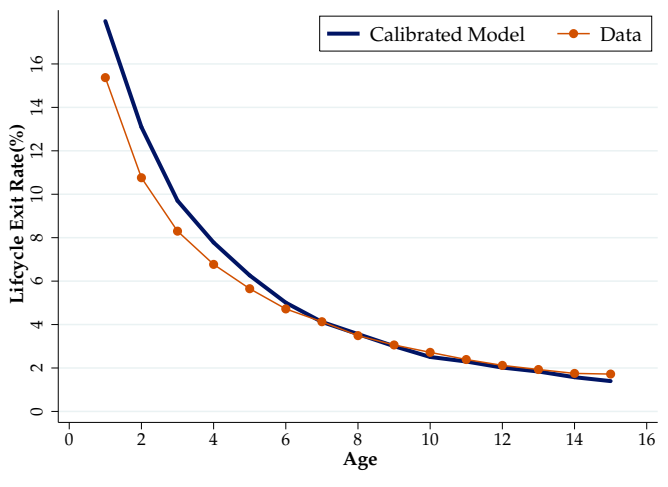
(b) is also quite good, even though in the data markups appear somewhat more linear with age than emerge from the model. Empirically, markups are increasing almost linearly by 1% each year. In the model, the rate of markup growth is slightly more concave.

Our model also makes precise predictions for the exit hazard. Our model predicts that the exit rate is declining in firm age. These declining exit rates by age reflect the fact that older firms have more product lines. Since the risk of product line destruction is independent across products, owning more products makes it progressively less likely that they will all be destroyed within a particular year. In Figure (4) we compare the model's predictions for exit rates by age to the data. To construct exit rates by age, we estimate a non-parametric Kaplan-Meier survival function by age for firms in the LBD. We select the cohort of firms born between 1980 and 1990, and follow them until 2015. We then take the exit rates to be the increments of the estimated survival functions. Each estimate is essentially the fraction of the sample that exit at age $a$ (though the estimator accounts for the truncation from ceasing to observe firms after 2015). Figure (4) shows that our model successfully replicates theses exit rates, despite the absence of exit moments in our calibration strategy.

### 3.4 Counterfactual: The Aggregate Impact of a Decline in Population Growth

Our theory highlights the tight relationship between population growth, the equilibrium rate of firm-creation and firms' average size. These patterns are very much visible in the data. In Figure 5 we display the time series of the rate of labor force growth (left panel) and the entry rate (middle panel). They track each other very closely. Quantitatively, the growth rate of the labor force has declined by roughly one percentage point since 1980. The entry rate has declined from about 14% in the 1980s to 10% in the 2015. Our theory highlights that this decline in population growth should increase average

Figure 4: Firm Exit Rates: Model and Data




Note: This figure presents a comparison of lifecycle exit rates between model and data. The exit rates in the data are taken from the increments in a Kaplan-Meier survival function estimated on all firms in the LBD born between 1980 and 1990. The model exit rates come from simulating a panel of $10^6$ firms and calculating the fraction of the panel that exit at yearly frequencies. Age of $a$ on the horizontal axis indicates that the firm exited between age $a-1$ and age $a$. $N = 70,000,000$, where this count has been rounded to accord with U.S. Census disclosure rules.

firm size because incumbents' expansion efforts are unaffected. Again, there is strong evidence for this trend in the data: average firm size has increased by about 15%. The average firm had roughly 20 employees in the 80s. Today, it has about 23 employees.

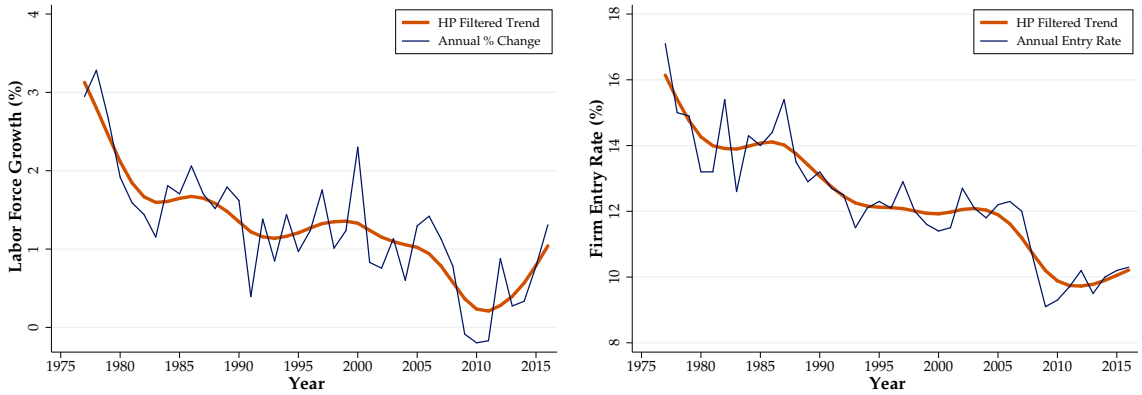
While the patterns shown in Figure 5 are qualitatively consistent with our theory, we now examine the implications implications of a 1% slow down in labor force growth quantitatively by comparing BGP's and holding all other parameters constant.

**Population Growth, Entry and Average Size**. In Figure 6 we display the entry rate and the average size as a function of the rate of population growth $\eta$. For comparison we also report the actual data from Figure 5. Figure 6 shows that the model can explain essentially the entirety of the decline in the entry rate and the increase in average size simply from the observed decline the rate of labor force growth.

**Population Growth and Markups**. In Figure 7 we report the aggregate implications of the slow-down of population growth on markups. In the left panel we show the estimated and the counter-factual life-cycle profile of markups. The decline in creative destruction reduces the extent to which markup growth by incumbents is kept in check. In addition, incumbents' incentives of engage in own-innovation increase as their face a lower risk of being replaced. Both these effects operate in the same direction: a decline in population growth increases markups, especially for old, large firms.

In the right panel, we show the cross-sectional distribution of markups. As implied by our theoretical results, the decline in population growth increases the average markup. Moreover, there is more mass on the maximum markup of $\sigma/(\sigma-1)$, which in our calibration is 33%. This reflects the fact that more products see enough innovation without being destroyed to reach the maximum markup

Figure 5: Labor Force Growth and Firm Entry

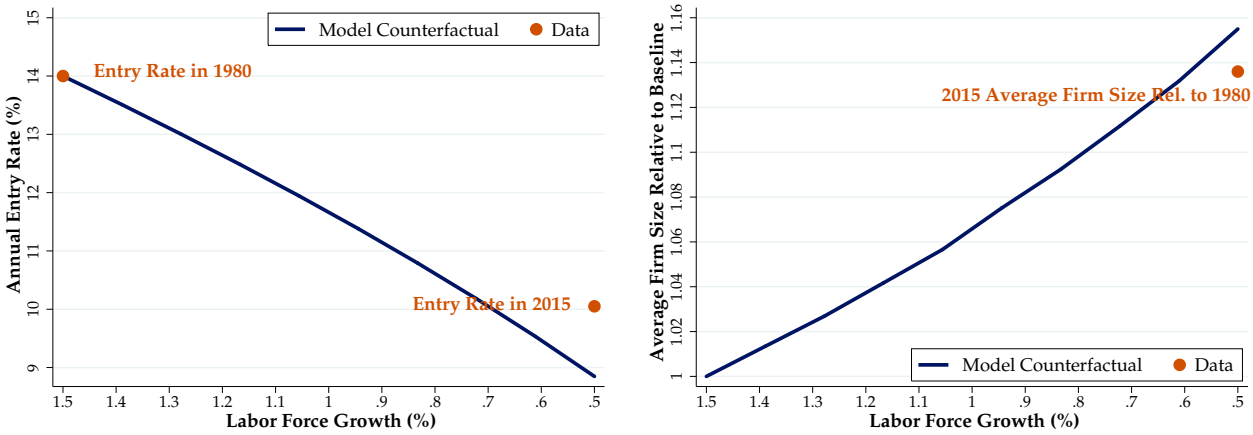



Note: The left panel shows the growth in the labor force of the U.S., with the raw series in blue and a HP-filtered trend component in orange. The data is sourced from the BLS, accessed through FRED. The right panel shows the entry rate of new firms, defined as the number of firms of age 0 divided by the total number of existing firms. The firm data is from the public-use Business Dynamics Statistics produced by the Center for Economic Studies at the U.S. Census Bureau.

Figure 6: Model Counterfactual for Entry and Firm Size

(a) Firm Entry Rates

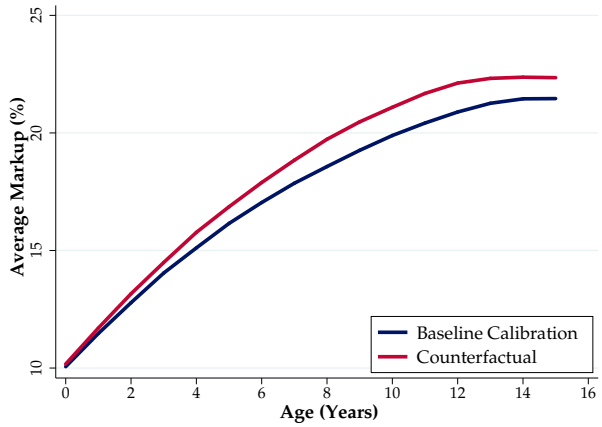
(b) Average Firm Size



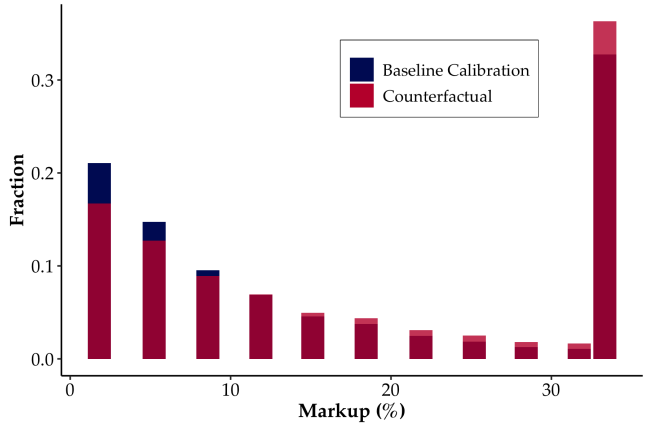

Note: Panel (a) in this Figure shows the prediction for the BGP entry rate in the calibrated model as we vary the labor force growth rate from 1.5% to 0.5%. All other calibrated parameters are held constant. Panel (b) shows the prediction for the BGP average firm size relative to the baseline calibrated model. Data for both the firm entry rate and the change in average firm size (shown in orange) are taken from the public use Business Dynamics Statistics produced by the Center for Economic Studies at the U.S. Census Bureau. Average firm size is defined in the data as total employment divided by total number of firms in the economy.

Figure 7: Model Counterfactual for Markups

(a) Average Markups By Age




(b) Product-level Markup Distribution




Note: Panel (a) in this Figure shows the BGP lifecycle growth in average markups at the firm level after a 1% decline in the labor force growth rate from the calibrated model (in red). The calibrated lifecycle for markups is shown in blue. Panel (b) shows the change in the product level markup distribution.

incumbents would like to charge on the product.

The average product markup increases by almost 2%, from 16.75% to 18.53%. This increase reflects two effects; the average markup by age increases, and firms are older on average, since they exit less frequently.

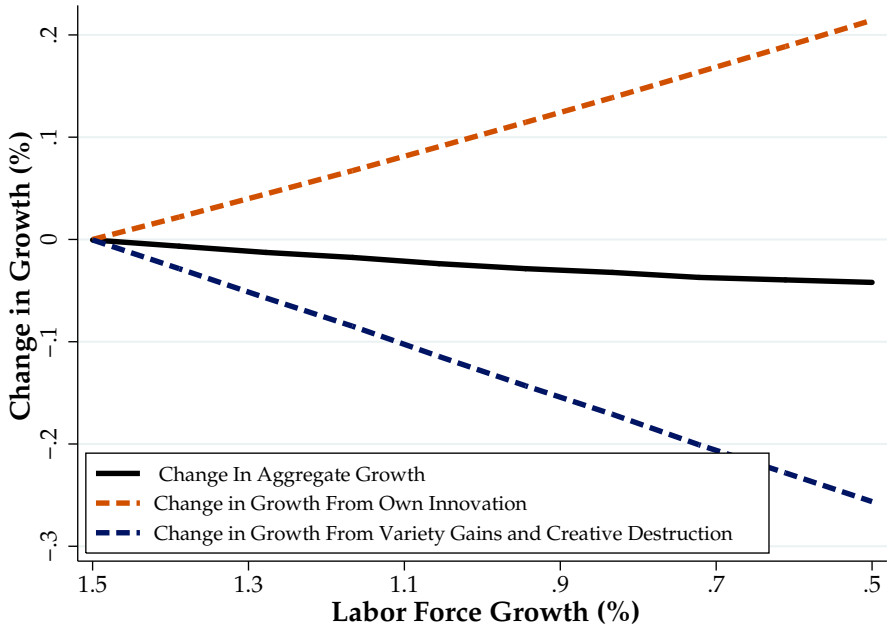
**Population Growth and Aggregate Growth**. Finally, we turn to the implications for aggregate productivity growth. Figure 8 shows the change in the equilibrium growth rate as a function of labor force growth $\eta$. It further decomposes the change in the growth rate into two components, which reflect changing incentives to innovate within product lines, and a second component which reflects slowing variety gains and creative destruction (the full equation is presented in Appendix B.5).

We find that slower population growth reduces the rate of productivity growth. Quantitatively, these effects are modest. We can see that this is due to the fact of strong offsetting effects from increased innovation by incumbents. While a decline in the rate of population growth reduces both creative destruction and growth in the number of varieties in the economy, it increases innovation efforts by incumbents to improve their own products. In our calibration, these effects almost entirely offset one another. More specifically, we find a decline in the growth rate of 0.04 percentage points when labor force growth declines by 1% from the calibrated baseline.

# 4 Conclusion

The U.S. economy has seen substantial changes in the past decades. Industry concentration and markups have been rising, productivity growth experienced a slowdown, entry rates have plum-

Figure 8: Model Counterfactual for Aggregate Growth




Note: This figure shows the change in the BGP growth rate from the baseline calibrated model obtained from varying the rate of labor force growth $\eta$ by 1%. Contributions from changes in own innovation are in orange, and the contribution from changes in variety gains and creative destruction are in blue, where these components are calculated according to equation (50) in Appendix B.5.

meted and measures of job reallocation have been systematically slower. These trends have sometimes been summarized as a "Decline in US Dynamism".

In this paper we showed that a single, empirically observable driver - the decline in the growth of the labor force - can qualitatively account for these seemingly disparate phenomena. We do so in the context of a frontier model of growth and firm dynamics, which features creative destruction by entrants and incumbents, the creation of new varieties, own-innovation by existing firms and endogenous market power. We show that this model makes tight predictions on how a slowdown in the rate of population growth affects firm dynamics, creative destruction, markups and growth. Lower population growth reduces entry, but does not affect incumbents' incentives for product creation. This implies that lower population growth unambiguously reduces creative destruction. We then show how this drop in creative destruction ripples through the economy. In particular, it increases markups and concentration and lowers the labor share.

To quantify the importance of the labor force channel, we calibrate the economy to data on sales and employment for the population of US firms. The model matches the observed life-cycle profiles of markups and sales and firms' exit hazard. In response to a reduction of the rate of labor force growth from 1.5% to 0.5% (as observed empirically between 1980 and 2005), the model generates essentially the entire decline in the entry rate from and the entire increase in average firm size. The model also implies markups to increase, especially in the right tail. The model's role in explaining

the slowdown in productivity growth is, however, limited. This is due to a powerful offsetting effect: the fall in creative destruction increases incumbent firms' incentives to raise quality to reap the benefits of higher markups. This response of incumbent innovation increases aggregate growth and highlights the importance of considering the interaction of different margins of innovation in general equilibrium models of endogenous growth.

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# Theory Appendix

## A The Special Case With $\sigma = 1$

In this Appendix, we consider the behavior of the model under Cobb-Douglass preferences for varieties. As discussed in Section 2.3, this case is particularly tractable and provides closed-form solutions for many objects of interest.

### A.1 Static Allocation and the Value Function

**Static Allocations.** The final good, which we take to be the numeraire, is a Cobb-Douglas composite of a continuum of differentiated varieties

$$\ln Y_t = \int_0^{N_t} \ln \left( \sum_{f \varepsilon S_{it}} y_{fit} \right) di, \tag{17}$$

where $N_t$ is the number of varieties available at time $t$. THence, sales per variety $i$ are given by

$$p_i y_i = \frac{Y_t}{N_t}$$

The mass of production workers is

$$L_{Pt} = \int_{i=0}^{N_t} l_{it} di = \frac{Y_t}{w_t} \left( \frac{1}{N_t} \int_{i=0}^{N_t} \mu_i^{-1} di \right).$$

Given that the final good is the numeraire, we have that

$$w_t = \exp \left( \frac{1}{N_t} \int_{i=0}^{N_t} \ln q_{it} di \right) \exp \left( \frac{1}{N_t} \int_{i=0}^{N_t} \ln \mu_{it}^{-1} di \right)$$

Hence,

$$Y_t = \frac{w_t}{\left( \frac{1}{N_t} \int_{i=0}^{N_t} \mu_i^{-1} di \right)} L_{Pt} = \exp \left( \frac{1}{N_t} \int_{i=0}^{N_t} \ln q_{it} di \right) \frac{\exp \left( \frac{1}{N_t} \int_{i=0}^{N_t} \ln \mu_{it}^{-1} di \right)}{\left( \frac{1}{N_t} \int_{i=0}^{N_t} \mu_i^{-1} di \right)} L_{Pt}$$

Let us denote the distribution of qualities $q$ by $F_t(q)$ and the distribution of markups $\mu$ by $G_t(\mu)$. Then

$$Q_t = \exp \left( \frac{1}{N_t} \int_{i=0}^{N_t} \ln q_{it} di \right) = \exp \left( \int_{\mathbb{R}_+} \ln q dF_t(q) \right)$$

and

$$\begin{aligned}
\exp\left(\frac{1}{N_t}\int_{i=0}^{N_t}\ln\mu_{it}^{-1}di\right) &= \exp\left(\int\ln\mu^{-1}dG_t(\mu)\right)=\exp\left(E\left[\ln\mu_{it}^{-1}\right]\right)\\
\frac{1}{N_t}\int_{i=0}^{N_t}\mu_i^{-1}di &= \int\mu^{-1}dG_t(\mu)=E\left[\mu_{it}^{-1}\right]
\end{aligned}$$

Then

$$\begin{aligned}
Y_t &= Q_t\mathcal{M}_tL_{Pt} \\
w_t &= \frac{Y_t}{L_{Pt}}\Lambda_t
\end{aligned} \tag{18}$$

where

$$\mathcal{M}_t=\frac{\exp\left(E\left[\ln\mu_{it}^{-1}\right]\right)}{E\left[\mu_{it}^{-1}\right]}\text{ and }\Lambda_t=E\left[\mu_{it}^{-1}\right].$$

Hence, along a BGP the growth rate of $Y_t$ is given by

$$g_Y=g_Q+\eta$$

as $L_{Pt}$ grows at rate $\eta$ and $\mathcal{M}_t$ is constant in a BGP. Wages grow at rate $g_Q$. Along a BGP it will also be the case that the number of varieties $N_t$ grows at the rate of the laborforce, i.e

$$g_N=\eta.$$

**The Value Function.** In the case of Cobb-Douglas demand, profits are given by

$$\pi_{it}=\left(1-\lambda^{-\Delta_i}\right)\frac{Y_t}{N_t}=\pi_t(\Delta_i).$$

Hence, the quality $q$ is not a state variable. Again, it is easy to verify that the value function is additively separable across products. Hence, consider a single product. The value function is given by

$$rV_t(\Delta)-\dot{V}_t(\Delta)=\pi_t(\Delta)-(\tau+\delta)V_t(\Delta)+I\left(V_t(\Delta+1)-V_t(\Delta)\right)+\max_x\left\{xV_t(1)-\frac{1}{\vartheta_x}x^{\zeta}w_t\right\}.$$

Conjecture that $V_t$ takes the form

$$V_t(\Delta)=\kappa w_t+\left(\theta-\beta\lambda^{-\Delta}\right)\frac{Y_t}{N_t},$$

where $\beta,\theta$ and $\kappa$ are to be determined. As $Y_t/N_t$ and $w_t$ grow at rate $g$ and $r-g=\rho$, we get that

$$(\rho+\tau+\delta)\left(\kappa w_t+\left(\theta-\beta\lambda^{-\Delta}\right)\frac{Y_t}{N_t}\right)=\pi_t(\Delta)+I\beta\lambda^{-\Delta}\left(\frac{\lambda-1}{\lambda}\right)\frac{Y_t}{N_t}+\max_x\left\{xV_t(1)-\frac{1}{\vartheta_x}x^{\zeta}w_t\right\}.$$

Hence,

$$(\rho+\tau+\delta)\left(\kappa w_t+\left(\theta-\beta\lambda^{-\Delta}\right)\frac{Y_t}{N_t}\right)=\left(1-\lambda^{-\Delta_i}\right)\frac{Y_t}{N_t}+I\beta\lambda^{-\Delta}\left(\frac{\lambda-1}{\lambda}\right)\frac{Y_t}{N_t}+\frac{\zeta-1}{\vartheta_x}x^{\zeta}w_t,$$

where

$$x=\left(\frac{\vartheta_x}{\zeta}\frac{V_1(x)}{w_t}\right)^{\frac{1}{\zeta-1}}. \tag{19}$$

Note $x$ is constant along a BGP. Hence,

$$\begin{aligned}(\rho+\tau+\delta)\kappa w_t &= \frac{\zeta-1}{\vartheta_x}x^{\zeta}w_t\\ (\rho+\tau+\delta)\left(\theta-\beta\lambda^{-\Delta}\right)\frac{Y_t}{N_t} &= \left(1-\lambda^{-\Delta_i}\right)\frac{Y_t}{N_t}+I\beta\lambda^{-\Delta}\left(\frac{\lambda-1}{\lambda}\right)\frac{Y_t}{N_t}\end{aligned}$$

This implies that

$$\begin{aligned}\kappa &= \frac{\frac{\zeta-1}{\vartheta_x}x^{\zeta}}{\rho+\tau+\delta}\\ \theta &= \frac{1}{\rho+\tau+\delta}\\ \beta &= \frac{1}{\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right)I}\end{aligned}$$

Hence,

$$V_t(\Delta)=\frac{\frac{\zeta-1}{\vartheta_x}x^{\zeta}w_t}{\rho+\tau+\delta}+\left(\frac{1}{\rho+\tau+\delta}-\frac{1}{\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right)I}\lambda^{-\Delta}\right)\frac{Y_t}{N_t}.$$

Note that (5) implies that

$$\frac{Y_t}{N_t}=\frac{w_tL_{Pt}}{N_t}\frac{1}{\Lambda}.$$

Hence,

$$\begin{aligned}V_t(1) &= \frac{\frac{\zeta-1}{\vartheta_x}x^{\zeta}w_t}{\rho+\tau+\delta}+\left(\frac{\left(\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right)I\right)-\frac{1}{\lambda}(\rho+\tau+\delta)}{(\rho+\tau+\delta)\left(\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right)I\right)}\right)\frac{Y_t}{N_t}\\ &= \frac{1}{\rho+\tau+\delta}\left[\frac{\zeta-1}{\vartheta_x}x^{\zeta}+\frac{\frac{\lambda-1}{\lambda}(\rho+\tau+\delta+I)}{\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right)I}\frac{L_{Pt}/N_t}{\Lambda}\right]w_t.\end{aligned} \tag{20}$$

**Free entry condition** The free entry condition is given by

$$V_t(1) = \frac{1}{\vartheta_z} w_t.$$

This has two implications.

(i) From (19) we know that

$$x = \left( \frac{\vartheta_x}{\zeta} \frac{V_1(x)}{w_t} \right)^{\frac{1}{\zeta-1}} = \left( \frac{1}{\zeta} \frac{\vartheta_x}{\vartheta_z} \right)^{\frac{1}{\zeta-1}} \tag{21}$$

(ii) Using (20) we get that

$$\frac{1}{\vartheta_z} = \frac{V_t(1)}{w_t} = \frac{1}{\rho+\tau+\delta} \left[ \frac{\zeta-1}{\vartheta_x} x^{\zeta} + \frac{\frac{\lambda-1}{\lambda}(\rho+\tau+\delta+I)}{\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right) I} \frac{L_{Pt}/N_t}{\Lambda} \right]. \tag{22}$$

Note that this equation determines $L_{Pt}/N_t$ in terms of parameters as $\Lambda$ is fully determined in Proposition 2.

## A.2 Population Growth and Creative Destruction (Proof of Proposition 1)

From (9) and (11) in the main text it follows directly that

$$\eta = \frac{\dot{N}_t}{N_t} = (1-\alpha)(x+z) - \delta.$$

Hence, (10) implies that

$$\tau = \alpha(x+z) = \frac{\alpha}{1-\alpha}(\eta+\delta).$$

This is (12) in Proposition 1. Finally, note that $x$ is given in (21) and that

$$z = \frac{\tau}{\alpha} - x = \frac{\eta+\delta}{1-\alpha} - x.$$

## A.3 Population Growth and Aggregate Growth (Proof of Proposition 3)

From (18), GDP per capita is given by:

$$y_t = \frac{Y_t}{L_t} = Q_t \mathcal{M}_t \frac{L_{Pt}}{L_t}.$$

Along the BGP, $\mathcal{M}_t$ and $\frac{L_{Pt}}{L_t}$ are constant. Hence, the growth rate of GDP pc is equal to the growth rate $Q_t$. Standard arguments imply that this growth rate is given by

$$g_Q = \frac{\dot{Q}_t}{Q_t} = \ln(\lambda)(\tau+I).$$

### A.4 The Distribution of Markups (Proof of Proposition 2)

The markup in product $i$ is given by

$$\mu_i = \mu(\Delta_i) = \lambda^{\Delta_i}$$

To determine the distribution of markups we hence only need to characterize the share of products with quality gap $\Delta$, $W_t(\Delta)$.

Consider first $W_t(1)$. The law of motion for the mass of products with a quality gap of one is given by

$$\begin{aligned}\frac{dW_t(1)}{dt} &= -(\delta + I + \tau) W_t(1) + \tau \sum_{\Delta \geq 1} W_t(\Delta) + \text{Flow of new Products} \\ &= -(\delta + I + \tau) W_t(1) + \tau N_t + \text{Flow of new Products},\end{aligned}$$

where $N_t$ is the total number of products. Because products are created at rate $(z + x^*)(1 - \alpha)$, the total mass of new products is given by

$$\text{Flow of new Products} = N_t (z + x^*)(1 - \alpha) = N_t \tau \left(\frac{1 - \alpha}{\alpha}\right).$$

Hence,

$$\begin{aligned}\frac{dW_t(1)}{dt} &= -(\delta + I + \tau) W_t(1) + \tau N_t + N_t \tau \left(\frac{1 - \alpha}{\alpha}\right) \\ &= -(\delta + I + \tau) W_t(1) + \frac{\tau N_t}{\alpha}\end{aligned}$$

Now consider the mass of products with $\Delta \geq 2$. The evolution of this mass satisfies the law of motion:

$$\frac{dW_t(\Delta)}{dt} = I W_t(\Delta - 1) - (\tau + I + \delta) W_t(\Delta).$$

Along the BGP $W_t(\Delta)$ grows at rate $\eta$. Hence, the above equations reduce to

$$\begin{aligned}\eta W_t(1) &= -(\delta + I + \tau) W_t(1) + \frac{\tau N_t}{\alpha} \\ \eta W_t(\Delta) &= I W_t(\Delta - 1) - (\tau + I + \delta) W_t(\Delta).\end{aligned}$$

Define $\nu_t(\Delta) \equiv W_t(\Delta) / N_t = \nu(\Delta)$ as the share of products with quality gap $\Delta$. Then

$$\begin{aligned}\eta \nu(1) &= -(\delta + I + \tau) \nu(1) + \frac{\tau}{\alpha} \\ \eta \nu(\Delta) &= I \nu(\Delta - 1) - (\tau + I + \delta) \nu(\Delta).\end{aligned}$$

Hence,

$$\begin{aligned}
\nu(1) &= \frac{1}{\alpha}\frac{\tau}{\delta+I+\tau+\eta} \\
\nu(\Delta) &= \frac{I}{\delta+I+\tau+\eta}\nu(\Delta-1).
\end{aligned}$$

It is easy to verify that

$$\nu(\Delta) = \frac{1}{\alpha}\frac{\tau}{I}\left(\frac{I}{\delta+I+\tau+\eta}\right)^{\Delta}. \tag{23}$$

Note that

$$\sum_{\Delta=1}^{\infty}\nu(\Delta) = \frac{1}{\alpha}\frac{\tau}{I}\sum_{\Delta=1}^{\infty}\left(\frac{I}{\delta+I+\tau+\eta}\right)^{\Delta} = 1$$

as required.

From (23) we can derive the distribution of markups. Note first that

$$\begin{aligned}
P(\Delta \leq d) &= \sum_{\Delta=1}^{d}\nu(\Delta) = \sum_{\Delta=1}^{d}\frac{1}{\alpha}\frac{\tau}{I}\left(\frac{I}{\delta+I+\tau+\eta}\right)^{\Delta} = 1-\left(\frac{I}{I+\frac{\eta+\delta}{1-\alpha}}\right)^{d} \\
&= 1-\left(\frac{1}{1+\frac{\eta+\delta}{(1-\alpha)I}}\right)^{d} \\
&= 1-e^{-\gamma(\eta)\times d}
\end{aligned}$$

where

$$\gamma(\eta) = \ln\left(1+\frac{\eta+\delta}{(1-\alpha)I}\right) \tag{24}$$

Hence (treating the distribution as continuous), the distribution of log markups is given by

$$P(\ln\mu \leq m) = P\left(\Delta \leq \frac{m}{\ln\lambda}\right) = 1-e^{-\frac{\gamma(\eta)}{\ln\lambda}m},$$

i.e. is an exponential distribution. The distribution of markups is therefore given by

$$\begin{aligned}
P(\mu \leq m) &= P(\ln\mu \leq \ln m) = 1-e^{-\frac{\gamma(\eta)}{\ln\lambda}\ln m} = 1-e^{\ln m^{-\frac{\gamma(\eta)}{\ln\lambda}}} \\
&= 1-m^{-\frac{\gamma(\eta)}{\ln\lambda}}.
\end{aligned}$$

where the pareto tail of the markup distribution is given in (24). The pareto tail is increasing in $\eta$ and decreasing in $I$.

## A.5 The Firm-Size Distribution

We first derive the firm-size distribution, i.e. the distribution of products across firms. Let $\tau + \delta$ denote the rate of losing a product and $x$ the rate of gaining a product and $z$ the flow rate of entry. Let $\omega_t(n)$ be the *mass* of firms with $n$ products at time $t$. Consider $n \geq 2$. Then

$$\dot{\omega}_t(n) = \underbrace{\omega_t(n-1)(n-1)x}_{\text{From } n-1 \text{ to} n \text{ products}} + \underbrace{\omega_t(n+1)(n+1)(\tau+\delta)}_{\text{From } n+1 \text{ to} n \text{ products}} - \underbrace{\omega_t(n)\, n(\tau+x+\delta)}_{\text{From } n \text{ to } n-1 \text{ or } n+1 \text{ products}}.$$

For $n = 1$ we have

$$\dot{\omega}_t(1) = z_t + \omega_t(2)\, 2(\tau+\delta) - \omega_t(1)(\tau+x+\delta).$$

Along the BGP the mass of firms grows at rate $\eta$. Intuitively: the distribution of firms across products is stationary and the number of products $N_t$ is increasing at rate $\eta$. Hence, the mass of firms is increasing at rate $\eta$. Hence, along the BGP we have

$$\dot{\omega}_t(n) = \eta\omega_t(n).$$

This implies

$$\begin{aligned}
\eta\omega_t(n) &= \omega_t(n-1)(n-1)x + \omega_t(n+1)(n+1)(\tau+\delta) - \omega_t(n)\, n(\tau+x+\delta) \\
\eta\omega_t(1) &= z_t + \omega_t(2)\, 2(\tau+\delta) - \omega_t(1)(\tau+x+\delta).
\end{aligned}$$

Rewriting yields

$$0 = \omega_t(n-1)(n-1)x + \omega_t(n+1)(n+1)(\tau+\delta) - \omega_t(n)\, n(\tau+x+\delta) - \omega_t(n)\eta \tag{25}$$

$$0 = z_t + \omega_t(2)\, 2(\tau+\delta) - \omega_t(1)(\tau+x+\delta) - \omega_t(1)\eta \tag{26}$$

Then

$$\omega_t(2) = \frac{\omega_t(1)(\tau+x+\delta+\eta) - z_t}{2(\tau+\delta)}$$

and

$$\omega_t(n+1) = \frac{\omega_t(n)\, n(\tau+x+\delta) + \omega_t(n)\eta - \omega_t(n-1)(n-1)x}{(n+1)(\tau+\delta)}$$

These are all homogeneous in $N_t$, the number of products. Let $\nu(n) = \frac{\omega_t(n)}{N_t}$ and $z = \frac{z_t}{N_t}$. Then

$$\nu(2) = \frac{\nu(1)(\tau+x+\delta+\eta) - z}{2(\tau+\delta)} \tag{27}$$

and

$$\nu(n+1) = \frac{\nu(n)\, n(\tau+x+\delta) + \nu(n)\eta - \nu(n-1)(n-1)x}{(n+1)(\tau+\delta)} \quad \text{for } n \geq 2 \tag{28}$$

Recall that $z$, $\tau$ and $x$ are determined in closed form in the equilibrium. Given $\nu(1)$, these equations

fully determine $[\nu(n)]_{n\geq 2}$. We can then pin down $\nu(1)$ from the requirement that

$$\sum_{n=1}^{\infty} \nu(n)\, n = \sum_{n=1}^{\infty} \frac{\omega_t(n)}{N_t} n = \frac{\sum_{n=1}^{\infty} \omega_t(n)\, n}{N_t} = 1. \tag{29}$$

Hence, equations (27), (28) and (29) fully determine the firm-size distribution $[\nu(n)]_{n\geq 1}$.

### A.6 The Allocation of Labor

**Solving for the allocation of production workers.** From the free entry condition in (22) we get that

$$\frac{1}{\vartheta_z} = \frac{1}{\rho+\tau+\delta}\left[\frac{\zeta-1}{\vartheta_x}x^{\zeta} + \frac{\frac{\lambda-1}{\lambda}(\rho+\tau+\delta+I)}{\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right)I}\frac{L_{Pt}/N_t}{\Lambda}\right].$$

This implies that

$$\frac{L_{Pt}}{N_t} = \Lambda \frac{\rho+\tau+\delta+\left(\frac{\lambda-1}{\lambda}\right)I}{\frac{\lambda-1}{\lambda}(\rho+\tau+\delta+I)}\left(\frac{\rho+\tau+\delta}{\vartheta_z} - \frac{\zeta-1}{\vartheta_x}x^{\zeta}\right).$$

Using that $\tau = \frac{\alpha}{1-\alpha}(\eta+\delta)$, we get that

$$\frac{L_{Pt}}{N_t} = \Lambda \frac{\lambda}{\lambda-1}\frac{\alpha\eta+\delta+(1-\alpha)\left(\rho+\left(\frac{\lambda-1}{\lambda}\right)I\right)}{(\alpha\eta+\delta+(1-\alpha)(\rho+I))}\left(\frac{(1-\alpha)\rho+\alpha\eta+\delta}{(1-\alpha)\vartheta_z} - \frac{\zeta-1}{\vartheta_x}x^{\zeta}\right)$$

Also, using that $x = \left(\frac{1}{\zeta}\frac{\vartheta_x}{\vartheta_z}\right)^{\frac{1}{\zeta-1}}$ (see (21)) we get that

$$\frac{\zeta-1}{\vartheta_x}x^{\zeta} = \frac{1}{\vartheta_z}(\zeta-1)\frac{\vartheta_z}{\vartheta_x}\left(\frac{1}{\zeta}\frac{\vartheta_x}{\vartheta_z}\right)^{\frac{\zeta}{\zeta-1}} = \frac{1}{\vartheta_z}\left(\frac{\zeta-1}{\zeta}\right)\left(\frac{1}{\zeta}\frac{\vartheta_x}{\vartheta_z}\right)^{\frac{1}{\zeta-1}}.$$

Hence,

$$\frac{L_{Pt}}{N_t} = \Lambda \frac{\lambda}{\lambda-1}\frac{\alpha\eta+\delta+(1-\alpha)\left(\rho+\left(\frac{\lambda-1}{\lambda}\right)I\right)}{(\alpha\eta+\delta+(1-\alpha)(\rho+I))}\frac{1}{\vartheta_z}\left(\frac{(1-\alpha)\rho+\alpha\eta+\delta}{(1-\alpha)} - \left(\frac{\zeta-1}{\zeta}\right)x\right)$$

Note that

$$\begin{aligned}\frac{\partial}{\partial\eta}\left[\frac{\alpha\eta+\delta+(1-\alpha)\left(\rho+\left(\frac{\lambda-1}{\lambda}\right)I\right)}{(\alpha\eta+\delta+(1-\alpha)(\rho+I))}\right] &= \frac{\alpha(\alpha\eta+\delta+(1-\alpha)(\rho+I)) - \left(\alpha\eta+\delta+(1-\alpha)\left(\rho+\left(\frac{\lambda-1}{\lambda}\right)I\right)\right)\alpha}{(\alpha\eta+\delta+(1-\alpha)(\rho+I))^2} \\ &= \frac{\alpha(1-\alpha)\frac{1}{\lambda}I}{(\alpha\eta+\delta+(1-\alpha)(\rho+I))^2} > 0.\end{aligned}$$

Hence,

$$\frac{\partial L_{P,t}/N_t}{\partial\eta} > 0,$$

because $\frac{\partial\Lambda}{\partial\eta} > 0$ and $\frac{\partial\frac{(1-\alpha)\rho+\alpha\eta+\delta}{(1-\alpha)}}{\partial\eta} > 0$.

**Total firm employment.** Total employment by firms is given by

$$L_t^{Firm} = L_t^P + L_t^X = L_t^P + N_t \frac{1}{\vartheta_x} x^{\zeta}.$$

Hence,

$$\begin{aligned}
\frac{L_t^{Firm}}{N_t} &= \Lambda \frac{\lambda}{\lambda - 1} \frac{\alpha\eta + \delta + (1-\alpha)\left(\rho + \left(\frac{\lambda-1}{\lambda}\right) I\right)}{(\alpha\eta + \delta + (1-\alpha)(\rho + I))} \frac{1}{\vartheta_z} \left( \frac{(1-\alpha)\rho + \alpha\eta + \delta}{(1-\alpha)} - \left(\frac{\zeta - 1}{\zeta}\right) x \right) + \frac{1}{\vartheta_x} x^{\zeta} \\
&= \frac{1}{\vartheta_z} \left[ \Lambda \frac{\lambda}{\lambda - 1} \frac{\alpha\eta + \delta + (1-\alpha)\left(\rho + \left(\frac{\lambda-1}{\lambda}\right) I\right)}{(\alpha\eta + \delta + (1-\alpha)(\rho + I))} \left( \frac{(1-\alpha)\rho + \alpha\eta + \delta}{(1-\alpha)} - \left(\frac{\zeta - 1}{\zeta}\right) x \right) + \frac{1}{\zeta} x \right] \quad (30)
\end{aligned}$$

## A.7 The Life-cycle of Sales

Note that sales is given by $py_{ft} = \frac{Y_t}{N_t} n_{ft}$, i.e. sales are proportional to the number of products. Let $p_n(a)$ be the prob that a firm has $n$ products at age $a$. This evolves according to

$$\dot{p}_n(a) = (n-1)\, x p_{n-1}(a) + (n+1)(\tau + \delta)\, p_{n+1}(a) - n(x + \tau + \delta)\, p_n(a).$$

Also, exit is an absorbing state, i.e.

$$\dot{p}_0(a) = (\tau + \delta)\, p_1(a)$$

This is the same equation as in Klette and Kortum (2004) but where the effective rate of creative destruction is $\tau + \delta$. In particular, the solution is as follows. Let

$$\gamma(a) = \frac{x\left(1 - e^{-(\tau+\delta-x)a}\right)}{\tau + \delta - x \times e^{-(\tau+\delta-x)a}}.$$

Then

$$\begin{aligned}
p_0(a) &= \frac{\tau + \delta}{x} \gamma(a) \\
p_1(a) &= (1 - p_0(a))(1 - \gamma(a)) \\
p_n(a) &= p_{n-1}(a)\, \gamma(a)
\end{aligned}$$

Now note the term $\tau + \delta - x$. We know that $\tau = \alpha(x + z)$. We also note that (along the BGP) $\frac{1-\alpha}{\alpha}\tau = \delta + \eta$. Hence,

$$\tau + \delta - x = \tau + \frac{1-\alpha}{\alpha}\tau - \eta - x = \frac{1}{\alpha}\tau - \eta - x = z - \eta. \quad (31)$$

Equation (31) implies a restriction on parameters. To make the firm-size distribution stationary, we need that $\tau + \delta - x < 0$. Hence, along the BGP with a stationary solution, we require $z > \eta$. If $\eta = 0$ this is always satisfied. With $\eta > 0$, the entry flow has to be large enough. Intuitively: if there is too little entry, incumbents are "grabbing" all the additional products, which are available because of the

larger population size. This makes older firms larger and larger (conditional on survival). Note that this terms also shows in the flow equations for the stationary distribution.

The *conditional* distribution of $n$ given age $a$ of surviving firms is given by $f_n(a) \equiv \frac{p_n(a)}{1-p_0(a)}$. Hence,

$$f_n(a) = (1-\gamma(a))\,\gamma(a)^{n-1}.$$

Hence, the expected log sales of a firm at time $t$ who if of age $a$ are given by

$$\begin{aligned} E[\ln S_t(n)|a] = \ln\frac{Y_t}{N_t} + E[\ln n|a] &= \ln\frac{Y_t}{N_t} + \sum_{n=1}^{\infty}\ln n f_n(a) \\ &= \ln\frac{Y_t}{N_t} + \left(\frac{1-\gamma(a)}{\gamma(a)}\right)\sum_{n=1}^{\infty}\ln n\gamma(a)^n. \end{aligned}$$

Note that this expression only depends on $\delta$, $\tau$ and $x$.

## A.8 The Life-cycle of Markups

Consider the life-cycle of markups. Let $p_\Delta(a)$ denote the probability of the product of a given firm having a quality gap $\Delta$ at age $a$. If the firm lost the product, we denote it by a quality gap of 0. Hence, 0 is an absorbing state.

$$\dot{p}_0(a) = (1-p_0(a))(\tau+\delta)$$

Also: firms always enter at 1. Hence, you can only leave state 1 so that

$$\dot{p}_1(a) = -p_1(a)(I+\tau+\delta).$$

Finally, for all $\Delta \geq 2$ we have

$$\dot{p}_\Delta(a) = p_{\Delta-1}(a)\,I - p_\Delta(a)(I+\tau+\delta)$$

The solution to this set of differential equations is given by

$$\begin{aligned} p_0(a) &= 1-e^{-(\tau+\delta)\times a} \\ p_{i+1}(a) &= \left(\frac{1}{i!}\right)I^i a^i\left(e^{-(I+\tau+\delta)a}\right) \text{ for } i \geq 0. \end{aligned}$$

The distribution of quality gaps conditional on survival is then

$$p^S_{i+1}(a) \equiv \frac{p_{i+1}(a)}{1-p_0(a)} = \left(\frac{1}{i!}\right)I^i a^i\left(e^{-Ia}\right).$$

As expected: this does not depend on $\tau$ nor $\delta$. This is a Poisson distribution with parameter $Ia$, so that $E[\Delta|a] = Ia$. Now note that $\mu = \lambda^\Delta$. Hence, the expected log markup by age is given by

$$E[\ln\mu|a] = \ln\lambda \times E[\Delta|a]. \tag{32}$$

Following the steps in Peters (2018), it can be shown that

$$E\left[\Delta|a\right] = 1 + I \times E\left[a_P|a_f\right],$$

where

$$\begin{aligned} E\left[a_P|a_f\right] &= \frac{\frac{1}{\tau+\delta}\left(1-e^{-(\tau+\delta)a}\right) - \frac{1}{x}e^{-(\tau+\delta)a}\left(1-e^{-xa}\right)}{1-e^{-(x+\tau+\delta)a}} \\ &+ \left(a - \frac{\frac{1}{\tau+\delta}\left(1-e^{-(\tau+\delta)a}\right) - \frac{1}{x}e^{-(\tau+\delta)a}\left(1-e^{-xa}\right)}{1-e^{-(x+\tau+\delta)a}}\right) \\ &\times \left(\frac{(\tau+\delta)\,e^{-xa} - xe^{-(\tau+\delta)a}}{x\left(1-e^{-(\tau+\delta-x)a}\right)}\right)\ln\left(\frac{\tau+\delta-x\times e^{-(\tau+\delta-x)a}}{\tau+\delta-x}\right). \end{aligned}$$

Note that $\lim_{a\to\infty} E\left[a_P|a\right] = \frac{1}{\tau+\delta}$, which is intuitive because product-firm pairs exit at rate $\tau+\delta$ which makes $\frac{1}{\tau+\delta}$ the expected age. This implies that

$$\lim_{a\to\infty} E\left[\ln\mu|a\right] = \ln\lambda\left(1+\frac{I}{\tau+\delta}\right) = \ln\lambda\left(1+\frac{(1-\alpha)\,I}{\alpha\eta+\delta}\right),$$

which shows that the expected markup for very old firms is declining in $\eta$ and increasing in $I$.

# B The General Case

In this section of the Appendix, we characterise the behaviour of the model in the general case of C.E.S preferences over varieties, such that $\sigma > 1$. In particular, we solve for the innovation choices of the firm, showing an important separability between quality and markup incentives. This ensures that the joint distribution of quality and markups is in fact simply the product of the two marginal distributions, greatly simplifying the equilibrium.

## B.1 The Value function

The state of the firm is given by $[\Delta_i, q_i]_{i=1}^n$. Conjecture that the value function is additive, i.e.

$$V_t\left([\Delta_i, q_i]_{i=1}^n\right) = \sum_{i=1}^n U_t\left([\Delta_i, q_i]\right) + n\phi_t$$

which consists of the rents from all the products, plus an expansion value of breaking into new markets. Then

$$V_t\left(\left[\Delta_j, q_j\right]_{j\neq i}\right) - V_t\left([\Delta_i, q_i]_{i=1}^n\right) = -U_t\left([\Delta_i, q_i]\right) - \phi_t$$

and

$$V_t\left(\left[\Delta_j, q_j\right]_{j\neq i}\left[\Delta_i+1, \lambda q_i\right]\right) - V_t\left(\left[\Delta_i, q_i\right]_{i=1}^n\right) = U_t\left(\left[\Delta+1_i, \lambda q_i\right]\right) - U_t\left(\left[\Delta_i, q_i\right]\right)$$

and

$$\int_q V_t\left(\left[\Delta_i, q_i\right]_{i=1}^n, 1, \lambda q\right) dF_t\left(q\right) - V_t\left(\left[\Delta_i, q_i\right]_{i=1}^n\right) = \int_q U_t\left(1, \lambda q\right) dF_t\left(q\right) + \phi_t = E_t\left[U_t\left(1, \lambda q\right)\right] + \phi_t$$

Hence we can write this as,

$$\begin{aligned}
&r\sum_{i=1}^n U_t\left(\left[\Delta_i, q_i\right]\right) - \sum_{i=1}^n(\dot{U}_t\left(\left[\Delta_i, q_i\right]\right) + (r+\tau+\delta)n\phi_t - n\dot{\phi}_t \\
&= \sum_{i=1}^n \pi_t\left(\left[\Delta_i, q_i\right]\right) - \sum_{i=1}^n(\tau+\delta)U_t\left(\left[\Delta_i, q_i\right]\right) - \\
&+\sum_{i=1}^n I\left[U_t\left(\left[\Delta+1_i, \lambda q_i\right]\right) - U_t\left(\left[\Delta_i, q_i\right]\right)\right] \\
&+\max_x\left\{nx\left[\delta_e E_t\left[U_t\left(1, \lambda q\right)\right] + (1-\delta_e)\mathbb{E}_\omega U_t\left(\delta, \omega\bar{q}_t\right) + \phi_t\right] - nc^X\left(\frac{x}{\phi_e}, 1\right)w_t\right\},
\end{aligned}$$

where $x = X/n$. Hence, we can write the RHS as

$$\begin{aligned}
&\sum_{i=1}^n\left[\pi_t\left(\left[\Delta_i, q_i\right]\right) - \tau U\left(\left[\Delta_i, q_i\right]\right) + I\left[U\left(\left[\Delta+1_i, \lambda q_i\right]\right) - U\left(\left[\Delta_i, q_i\right]\right)\right]\right] \\
&+n\max_x\left\{x\left[\delta_e\mathbb{E}_t\left[U_t\left(1, \lambda q\right)\right] + (1-\delta_e)\mathbb{E}_\omega U_t\left(\delta, \omega\bar{q}_t\right) + \phi_t\right] - c^X\left(\frac{x}{\phi_e}, 1\right)w_t\right\}
\end{aligned}$$

Hence, the value function is additive and we can for focus on a single product, which is described by

$$rU_t\left(\left[\Delta_i, q_i\right]\right) - \dot{U}_t\left(\left[\Delta_i, q_i\right]\right) = \pi_t\left(\left[\Delta_i, q_i\right]\right) - (\tau+\delta)U\left(\left[\Delta_i, q_i\right]\right) + I\left[U\left(\left[\Delta+1_i, \lambda q_i\right]\right) - U\left(\left[\Delta_i, q_i\right]\right)\right] \tag{33}$$

and separately solve for the expansion term

$$(r+\tau+\delta)n\phi_t - n\dot{\phi}_t = n\max_x\left\{x\left[\delta_e\mathbb{E}_t\left[U_t\left(1, \lambda q\right)\right] + (1-\delta_e)\mathbb{E}_\omega U_t\left(\delta, \omega\bar{q}_t\right) + \phi_t\right] - c^X\left(\frac{x}{\phi_e}, 1\right)w_t\right\}$$

**Solving for** $U_t\left(\left[\Delta_i, q_i\right]\right)$. Conjecture that the value function of a particular product takes the following form

$$U_t\left(\left[\Delta, q\right]\right) = \frac{u\left(\Delta\right)}{g(\sigma-1)+\rho+\tau+\delta}\frac{q^{\sigma-1}Y_t}{E[\mu^{1-\sigma}]N_tQ_t^{\sigma-1}}. \tag{34}$$

We need to determine the function $u\left(.\right)$ . Note that $Y_t$ is growing at rate $g+\eta$ and $Q_t^{\sigma-1}$ is growing at rate $g(\sigma-1)$. As such, along the BGP we have

$$\dot{U}_t\left(\left[\Delta, q\right]\right) = g(2-\sigma)U_t\left(\left[\Delta, q\right]\right) \tag{35}$$

So for a fixed $q$, whether profits are shrinking or rising with growth depends on whether $\sigma \lesseqgtr 2$. Now, using log preferences we have

$$r = g + \rho$$

and combining this with (35) and (33) gives:

$$\left(g(\sigma-1)+\rho+\tau+\delta\right) U_t\left([\Delta, q]\right) \quad = \quad \pi_t\left([\Delta, q]\right) + I\left[U\left([\Delta+1, \lambda q]\right) - U\left([\Delta, q]\right)\right]$$

Using our guess, we have

$$\begin{aligned} U_t\left([\Delta+1, \lambda q]\right) - U_t\left([\Delta, q]\right) &= \left(\frac{u\left(\Delta+1\right)}{g(\sigma-1)+\rho+\tau+\delta}\left(\lambda q\right)^{\sigma-1}\right)\frac{Y_t}{E[\mu^{1-\sigma}]N_t Q_t^{\sigma-1}} \\ &- \left(\frac{u\left(\Delta\right)}{g(\sigma-1)+\rho+\tau+\delta} q^{\sigma-1}\right)\frac{Y_t}{E[\mu^{1-\sigma}]N_t Q_t^{\sigma-1}} \\ &= \frac{u\left(\Delta+1\right)\left(\lambda\right)^{\sigma-1} - u\left(\Delta\right)}{g(\sigma-1)+\rho+\tau+\delta}\frac{q^{\sigma-1}Y_t}{E[\mu^{1-\sigma}]N_t Q_t^{\sigma-1}}. \end{aligned}$$

Recall that

$$Y_t = \left[\frac{E\left[\mu_i^{1-\sigma}\right]}{E\left[\mu_i^{-\sigma}\right]}\right] w_t L_t^P$$

Define $\Lambda \equiv E\left[\mu_i^{-\sigma}\right]^{-1}$. So then

$$\frac{Y_t}{w_t} = E\left[\mu_i^{1-\sigma}\right]\Lambda\frac{L_t^P}{N_t}N_t \tag{36}$$

or

$$w_t = \frac{Y_t}{E\left[\mu_i^{1-\sigma}\right]N_t}\left(\Lambda\frac{L_t^P}{N_t}\right)^{-1}$$

where $\frac{L_t^P}{N_t}$ is a key endogenous ratio that we need to solve for on the BGP.

Then

$$\left(g(\sigma-1)+\rho+\tau+\delta\right) U_t\left([\Delta, q]\right) \quad = \quad \pi_t\left([\Delta, q]\right) + I\frac{u\left(\Delta+1\right)\left(\lambda\right)^{\sigma-1} - u\left(\Delta\right)}{g(\sigma-1)+\rho+\tau+\delta}\frac{q^{\sigma-1}Y_t}{E[\mu^{1-\sigma}]N_t Q_t^{\sigma-1}}. \tag{37}$$

Substituting for $\pi_t\left([\Delta, q]\right)$ and $U_t\left([\Delta, q]\right)$ we get that

$$\left(g(\sigma-1)+\rho+\tau+\delta\right) U_t\left([\Delta, q]\right) \quad = \quad \pi_t\left([\Delta, q]\right) + I\frac{u\left(\Delta+1\right)\left(\lambda\right)^{\sigma-1} - u\left(\Delta\right)}{g(\sigma-1)+\rho+\tau+\delta}\frac{q^{\sigma-1}Y_t}{E[\mu^{1-\sigma}]N_t Q_t^{\sigma-1}}. \tag{38}$$

The optimality condition for $I$ is given by

$$\frac{u(\Delta+1)(\lambda)^{\sigma-1}-u(\Delta)}{g(\sigma-1)+\rho+\tau+\delta}\frac{q^{\sigma-1}Y_t}{E[\mu^{1-\sigma}]N_tQ_t^{\sigma-1}}=\zeta\varrho_I\frac{q^{\sigma-1}}{Q_t^{\sigma-1}}I^{\zeta-1}w_t$$

This yields

$$I(\Delta)=\left(\frac{u(\Delta+1)(\lambda)^{\sigma-1}-u(\Delta)}{g(\sigma-1)+\rho+\tau+\delta}\frac{Y_t}{E[\mu^{1-\sigma}]N_tw_t}\frac{1}{\varrho_I\zeta}\right)^{\frac{1}{\zeta-1}}=\left(\frac{u(\Delta+1)(\lambda)^{\sigma-1}-u(\Delta)}{g(\sigma-1)+\rho+\tau+\delta}\frac{1}{\varrho_I\zeta}\Lambda\frac{L_t^P}{N_t}\right)^{\frac{1}{\zeta-1}}. \tag{39}$$

where the second equality uses (36). Hence,

$$\max_I\left\{I\frac{u(\Delta+1)(\lambda)^{\sigma-1}-u(\Delta)}{g(\sigma-1)+\rho+\tau+\delta}\frac{q^{\sigma-1}Y_t}{E[\mu^{1-\sigma}]N_tQ_t^{\sigma-1}}-\varrho_I\frac{q^{\sigma-1}}{Q_t^{\sigma-1}}I^{\zeta}w_t\right\}=(\zeta-1)\varrho_II(\Delta)^{\zeta}w_t\frac{q^{\sigma-1}}{Q_t^{\sigma-1}}.$$

Substituting this and our guess into (37) yields

$$u(\Delta)\frac{q^{\sigma-1}Y_t}{E[\mu^{1-\sigma}]N_tQ_t^{\sigma-1}} = h(\Delta)q^{\sigma-1}\frac{1}{\mathbb{E}_t[\mu^{1-\sigma}]}\frac{Y_t}{Q_t^{\sigma-1}}\frac{1}{N_t}+(\zeta-1)\varrho_II(\Delta)^{\zeta}w_t\frac{q^{\sigma-1}}{Q_t^{\sigma-1}},$$

where $h(\Delta)$ is given as

$$h(\Delta)=\left(1-\frac{1}{\mu(\Delta)}\right)\mu(\Delta)^{1-\sigma}.$$

Rearranging terms and using (36) this yields

$$u(\Delta) = h(\Delta)+(\zeta-1)\varrho_II(\Delta)^{\zeta}\left(\Lambda\frac{L_t^P}{N_t}\right)^{-1}.$$

Substituting the optimal solution for $I$ in (39) yields

$$u(\Delta) = h(\Delta)+\frac{(\zeta-1)\varrho_I}{((g(\sigma-1)+\rho+\tau+\delta)\varrho_I\zeta)^{\frac{\zeta}{\zeta-1}}}\left(u(\Delta+1)(\lambda)^{\sigma-1}-u(\Delta)\right)^{\frac{\zeta}{\zeta-1}}\left(\Lambda\frac{L_t^P}{N_t}\right)^{\frac{1}{\zeta-1}}. \tag{40}$$

This is a difference equation in $u(\Delta)$ given parameters and the general equilibrium statistic $\Lambda\frac{L_t^P}{N_t}$. Given $\{u(\Delta)\}_\Delta$, firms' optimal innovation rate is given by

$$I(\Delta)=\left(\frac{u(\Delta)-h(\Delta)}{(\zeta-1)\varrho_I}\Lambda\frac{L_t^P}{N_t}\right)^{1/\zeta}. \tag{41}$$

Note crucially that there is no dependence on a product's quality; innovation only depends on the gap between the quality of the firm and the next best producer. In Section B.6 we show that there is a unique solution $I(\Delta)$ to (40) and (41).

### B.2 Free Entry

The free entry condition can be written as

$$w_t = \vartheta_E \left(\delta_e \mathbb{E}_t \left[U_t\left(1, \lambda q\right)\right] + (1-\delta_e)\mathbb{E}_\omega U_t\left(\Delta_0, \omega \bar{q}_t\right) + \phi_t\right) \tag{42}$$

Now note that (34) implies that

$$\mathbb{E}_t \left[U_t\left(1, \lambda q\right)\right] = \mathbb{E}_t \left[\frac{u\left(1\right)}{g(\sigma-1)+\rho+\tau+\delta_0} \frac{\left(\lambda q\right)^{\sigma-1} Y_t}{E[\mu^{1-\sigma}]N_t Q_t^{\sigma-1}}\right] = \frac{u\left(1\right)}{g(\sigma-1)+\rho+\tau+\delta_0} \frac{\lambda^{\sigma-1} Y_t}{E[\mu^{1-\sigma}]N_t}$$

and

$$\mathbb{E}_t \left[U_t\left(\Delta_0, \omega \bar{q}_t\right)\right] = \frac{u\left(\Delta_0\right)}{g(\sigma-1)+\rho+\tau+\delta_0} \frac{\mathbb{E}_t\left[\omega^{\sigma-1}\right] Y_t}{E[\mu^{1-\sigma}]N_t} = \frac{u\left(\Delta_0\right)}{g(\sigma-1)+\rho+\tau+\delta_0} \frac{\overline{\omega} Y_t}{E[\mu^{1-\sigma}]N_t}.$$

Using (45) we get that

$$w_t = \vartheta_E \left[\frac{\delta_e u\left(1\right) \lambda^{\sigma-1} + (1-\delta_e)\, u\left(\Delta_0\right) \overline{\omega}}{g(\sigma-1)+\rho+\tau+\delta_0} + \Phi\right] \frac{Y_t}{E[\mu^{1-\sigma}]N_t}.$$

Substituting (36), we have

$$\frac{1}{\Lambda \frac{L_t^p}{N_t}} = \vartheta_E \left[\frac{\delta_e u\left(1\right) \lambda^{\sigma-1} + (1-\delta_e)\, u\left(\Delta_0\right) \overline{\omega}}{g(\sigma-1)+\rho+\tau+\delta_0} + \Phi\right] \tag{43}$$

### B.3 Incumbent Expansion

We now characterize the solution to the expansion value of the firm. Recall this is given by

$$(r+\tau+\delta_0)\phi_t - \dot{\phi}_t = \max_x \left\{x\left[\delta_e \mathbb{E}_t\left[U_t\left(1, \lambda q\right)\right] + (1-\delta_e)\mathbb{E}_\omega U_t\left(\Delta_0, \omega \bar{q}_t\right) + \phi_t\right] - c^X\left(x, 1\right) w_t\right\} \tag{44}$$

We guess and verify that

$$\phi_t = \frac{Y_t}{E[\mu^{1-\sigma}]N_t}\Phi, \tag{45}$$

where $\Phi$ is a constant we need to solve for. This guess implies that on the BGP $\dot{\phi}_t = g\phi_t$. Now the optimality condition for expansion is given by

$$\delta_e \mathbb{E}_t\left[U_t\left(1, \lambda q\right)\right] + (1-\delta_e)\mathbb{E}_\omega U_t\left(\Delta_0, \omega \bar{q}_t\right) + \phi_t = \zeta \varrho_x \left(x^*\right)^{\zeta-1} w_t. \tag{46}$$

Using (42) we can write (46) as

$$\vartheta_E^{-1} w_t = \zeta \varrho_x \left(x^*\right)^{\zeta-1} w_t.$$

Hence, the optimal expansion rate $x^*$ is given by

$$x^* = \left(\frac{1}{\vartheta_E \zeta \varrho_x}\right)^{\frac{1}{\zeta-1}}. \tag{47}$$

Note that $x^*$ is only a function of parameters - not of endogenous variables. This implies that

$$\max_x \left\{ x\left[\delta_e \mathbb{E}_t\left[U_t\left(1, \lambda q\right)\right] + (1-\delta_e)\mathbb{E}_\omega U_t\left(\Delta_0, \omega \bar{q}_t\right) + \phi_t\right] - c^X\left(x, 1\right) w_t \right\} = \left(\zeta - 1\right) \varrho_x \left(x^*\right)^\zeta w_t.$$

Hence,

$$\phi_t = \frac{\left(\zeta-1\right)\varrho_x\left(x^*\right)^\zeta}{r+\tau+\delta_0-g} w_t = \frac{\left(\zeta-1\right)\varrho_x\left(x^*\right)^\zeta}{\rho+\tau+\delta_0} w_t.$$

where the second equality uses the Euler equation $r = \rho + g$. Using (36), we can write this equation as

$$\phi_t = \frac{\left(\zeta-1\right)\varrho_x\left(x^*\right)^\zeta}{\rho+\tau+\delta_0} \frac{1}{\Lambda \frac{L_t^P}{N_t}} \frac{Y_t}{E\left[\mu_i^{1-\sigma}\right] N_t} = \Phi \frac{Y_t}{E\left[\mu_i^{1-\sigma}\right] N_t} \tag{48}$$

which is of the form conjectured in (45).

### B.4 Proof of Proposition 2: The Stationary Distribution of Markups

The markup in product $i$ is given by

$$\mu_i = \mu(\Delta_i) = \min\left\{\lambda^{\Delta_i}, \frac{\sigma}{\sigma-1}\right\} = \begin{cases} \lambda^{\Delta_i} & \text{if } \Delta < \frac{\ln\left(\frac{\sigma}{\sigma-1}\right)}{\ln\lambda} \equiv \overline{\Delta} \\ \frac{\sigma}{\sigma-1} & \text{if } \Delta \geq \frac{\ln\left(\frac{\sigma}{\sigma-1}\right)}{\ln\lambda} \equiv \overline{\Delta} \end{cases}$$

To determine the distribution of markups we hence only need to characterize the share of products with quality gap $\Delta$, $W_t\left(\Delta\right)$.

Consider first $W_t\left(1\right)$. The law of motion for the mass of products with a quality gap of one is given by

$$\frac{dW_t\left(1\right)}{dt} = -\left(\delta + I + \tau\right) W_t\left(1\right) + \tau \sum_{\Delta \geq 1} W_t\left(\Delta\right) = -\left(\delta + I + \tau\right) W_t\left(1\right) + \tau N_t$$

where $N_t$ is the total number of products. Now consider the mass of products with $2 \leq \Delta < \overline{\Delta}$. The evolution of this mass satisfies the law of motion:

$$\frac{dW_t\left(\Delta\right)}{dt} = I W_t\left(\Delta - 1\right) - \left(\tau + I + \delta\right) W_t\left(\Delta\right)$$

Finally, consider the states $\Delta \geq \overline{\Delta}$. Because markups are the same in all products with $\Delta \geq \overline{\Delta}$, let $\Omega_t = \sum_{\Delta \geq \overline{\Delta}} W_t\left(\Delta\right)$ be the mass of products with a gap of at least $\overline{\Delta}$. In the interval $[t, t+h]$ this mass

evolves according to

$$\Omega_{t+h} = \Omega_t - h(\tau+\delta)\Omega_t + hIW_t\left(\overline{\Delta}-1\right) + \text{Flow of new Products}$$

as all new products start with the unconstrained markups. Because products are created at rate $(z+x^*)(1-\delta_e)$, the total mass of new products is given by

$$\text{Flow of new Products} = N_t h(z+x^*)(1-\delta_e) = N_t \tau h\left(\frac{1-\delta_e}{\delta_e}\right).$$

Hence,

$$\frac{d\Omega_t}{dt} = -(\tau+\delta)\Omega_t + IW_t\left(\overline{\Delta}-1\right) + N_t\tau\left(\frac{1-\delta_e}{\delta_e}\right).$$

Along the BGP $W_t(\Delta)$ and $\Omega_t$ grow at rate $\eta$. Hence, the above three equations reduce

$$\begin{aligned}
\eta W_t(1) &= -(\delta+I+\tau)W_t(1) + \tau N_t \\
\eta W_t(\Delta) &= IW_t(\Delta-1) - (\tau+I+\delta)W_t(\Delta) \\
\eta\Omega_t &= -(\tau+\delta)\Omega_t + IW_t\left(\overline{\Delta}-1\right) + N_t\tau\left(\frac{1-\delta_e}{\delta_e}\right).
\end{aligned}$$

Define $\nu_t(\Delta) \equiv W_t(\Delta)/N_t = \nu(\Delta)$ as the share of products with quality gap $\Delta$ and $\omega \equiv \Omega_t/N_t$. Then

$$\begin{aligned}
\eta\nu(1) &= -(\delta+I+\tau)\nu(1) + \tau \\
\eta\nu(\Delta) &= I\nu(\Delta-1) - (\tau+I+\delta)\nu(\Delta) \\
\eta\omega &= -(\tau+\delta)\omega + I\nu\left(\overline{\Delta}-1\right) + \tau\left(\frac{1-\delta_e}{\delta_e}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\nu(1) &= \frac{\tau}{\delta+I+\tau+\eta} \\
\nu(\Delta) &= \frac{I}{\delta+I+\tau+\eta}\nu(\Delta-1) \\
\omega &= \frac{I}{\delta+\tau+\eta}\nu\left(\overline{\Delta}-1\right) + \frac{\tau}{\delta+\tau+\eta}\left(\frac{1-\delta_e}{\delta_e}\right)
\end{aligned}$$

It is easy to verify that

$$\nu(\Delta) = \frac{\tau}{I}\left(\frac{I}{\delta+I+\tau+\eta}\right)^{\Delta}.$$

Now note that

$$\tau = \frac{\delta_e}{1-\delta_e}(\eta+\delta).$$

Then we can write

$$\begin{aligned}\nu(\Delta) &= \frac{\tau}{I}\left(\frac{I}{\delta+I+\tau+\eta}\right)^{\Delta} = \frac{\frac{\delta_e}{1-\delta_e}(\eta+\delta)}{I}\left(\frac{I}{\delta+I+\frac{\delta_e}{1-\delta_e}(\eta+\delta)+\eta}\right)^{\Delta} \\ &= \delta_e\theta\left(\frac{1}{1+\theta}\right)^{\Delta}\end{aligned}$$

where

$$\theta \equiv \frac{\eta+\delta}{(1-\delta_e)I}$$

.

Similarly,

$$\begin{aligned}\omega &= \frac{I}{\delta+\tau+\eta}\left[\frac{\tau}{I}\left(\frac{I}{\delta+I+\tau+\eta}\right)^{\overline{\Delta}-1}\right] + \frac{\tau}{\delta+\tau+\eta}\left(\frac{1-\delta_e}{\delta_e}\right) \\ &= \delta_e\left(\frac{1}{1+\theta}\right)^{\overline{\Delta}-1} + (1-\delta_e)\end{aligned}$$

Finally, note that

$$\begin{aligned}\sum_{\Delta<\overline{\Delta}}\nu(\Delta)+\omega &= \sum_{\Delta<\overline{\Delta}}\delta_e\theta\left(\frac{1}{1+\theta}\right)^{\Delta} + \delta_e\left(\frac{1}{1+\theta}\right)^{\overline{\Delta}-1} + (1-\delta_e) \\ &= \delta_e\theta\sum_{\Delta<\overline{\Delta}}\left(\frac{1}{1+\theta}\right)^{\Delta} + \delta_e\left(\frac{1}{1+\theta}\right)^{\overline{\Delta}-1} + (1-\delta_e).\end{aligned}$$

Now

$$\sum_{\Delta<\overline{\Delta}}\left(\frac{1}{1+\theta}\right)^{\Delta} = \sum_{\Delta=1}^{\overline{\Delta}-1}\left(\frac{1}{1+\theta}\right)^{\Delta} = \left(\frac{1}{1+\theta}\right)\frac{1-\left(\frac{1}{1+\theta}\right)^{\overline{\Delta}-1}}{1-\left(\frac{1}{1+\theta}\right)} = \frac{1}{\theta}\left(1-\left(\frac{1}{1+\theta}\right)^{\overline{\Delta}-1}\right).$$

Hence, $\sum_{\Delta<\overline{\Delta}}\nu(\Delta)+\omega = 1$ as required.

### B.5 Aggregate Growth

From the expression for wages, we have

$$g = \frac{1}{\sigma-1}\frac{dlogN_t}{dt} + \frac{1}{\sigma-1}dlog\mathbb{E}_t q^{\sigma-1}$$

$$= \frac{1}{\sigma-1}\eta + \frac{1}{\sigma-1}\frac{dlog\mathbb{E}_t q^{\sigma-1}}{dt}$$

Now consider how this object evolves through a discrete time approximation for a small interval $Dt$. First, assume that average innovation rates are stable on the BGP (we verify that this is so below). Examining (33), we see that optimal innovation rate $I$ of the firm for a product line is only a function of $\Delta$, and not $q$. Then let $I^*$ given as $I^* = \int_\Delta I(\Delta)\, dF(\Delta)$. We have

$$\mathbb{E}_t q^{\sigma-1} = \int_q q^{\sigma-1} dF_t(q^{\sigma-1})$$

$$= \underbrace{(1-e^{-(1-\delta_e)zDt})\bar{\omega}\int q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})}_{entrant\ new\ product} + \underbrace{(1-e^{-(1-\delta_e)x^*Dt})\bar{\omega}\int q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})}_{incumbent\ new\ product} +$$

$$+ \underbrace{(1-e^{-\delta_e zDt})\int \lambda^{\sigma-1}q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})}_{entrant\ existing\ product} + \underbrace{(1-e^{-\delta_e x^*Dt})\int \lambda^{\sigma-1}q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})}_{incumbent\ existing\ product}$$

$$\underbrace{(1-\delta_e e^{-I^*Dt})\int \lambda^{\sigma-1}q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})}_{incumbent\ own\ innovation} + \underbrace{\Sigma\int q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})}_{nothing}$$

where

$$\Sigma = 1-(1-e^{-(1-\delta_e)zDt}) - (1-e^{-(1-\delta_e)x^*Dt}) - (1-e^{-\delta_e zDt}) - (1-e^{-\delta_e x^*Dt}) - (1-\delta_e e^{-I^*Dt})$$

Aa first order expansion of each of these terms and rearranging yields

$$\frac{\int q^{\sigma-1}dF_t(q^{\sigma-1}) - \int q^{\sigma-1}dF_{t-Dt}(q)}{Dt} = [((1-\delta_e)(z_e+x^*))(\bar{\omega}-1)])\int q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})$$
$$+ \left[(\delta_e(z_e+x^*)+I^*)(\lambda^{\sigma-1}-1)\right])\int q^{\sigma-1}dF_{t-Dt}(q^{\sigma-1})$$

And then take $Dt$ small to get

$$\frac{d\mathbb{E}_t q^{\sigma-1}}{dt} = \left[(1-\delta_e)(z_e+x^*)(\bar{\omega}-1) + (\delta_e(z_e+x^*)+I^*)(\lambda^{\sigma-1}-1)\right]\mathbb{E}_t q^{\sigma-1} \tag{49}$$

Letting $\delta_e(z_e+x^*) = \tau$, we get that the rate of growth is given by

$$\begin{aligned} g &= \frac{1}{\sigma-1}\eta + \frac{1}{\sigma-1}\left[(1-\delta_e)(z^*+x^*)(\bar{\omega}-1) + (\tau+I^*)(\lambda^{\sigma-1}-1)\right] \\ &= \frac{1}{\sigma-1}\eta + \frac{1}{\sigma-1}\left[\left(\frac{1-\delta_e}{\delta_e}\right)\tau(\bar{\omega}-1) + (\tau+I^*)(\lambda^{\sigma-1}-1)\right]. \end{aligned} \tag{50}$$

Given $\tau$, the only endogenous variable in this equation is $I^*$. We now proceed to construct a continuous mapping on the real line that maps $I^*$ into the decisions of firms to innovate, and show the

existence of a fixed point.

### B.6 Proof that there is a unique solution for $I(\Delta)$ defined in (41)

Consider a firm with quality gap $\Delta \geq \overline{\Delta} = \frac{\ln \frac{\sigma}{\sigma-1}}{\ln \lambda}$. This firm will set a markup of $\frac{\sigma}{\sigma-1}$. Hence,

$$h(\Delta) = \left(\frac{1}{\sigma}\right)\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} = \overline{h} \text{ for } \Delta \geq \overline{\Delta}.$$

Moreover, conditional on survival, it will never set a different markup. Because we restrict our analysis to value function, which only depend on pay-off state variables, we know that $u(\Delta) = \overline{u}$ for $\Delta \geq \overline{\Delta}$. Hence, (40) and (41) imply that (for $\Delta \geq \overline{\Delta}$)

$$I = (C\overline{u})^{\frac{1}{\zeta-1}}$$

and

$$\overline{u} = \overline{h} + (\zeta - 1)\,\varrho_I C^{\frac{\zeta}{\zeta-1}} \left(\Lambda \frac{L_t^P}{N_t}\right)^{-1} \overline{u}^{\frac{\zeta}{\zeta-1}}, \tag{51}$$

where

$$C \equiv \frac{(\lambda^{\sigma-1} - 1)}{(g(\sigma-1) + \rho + \tau + \delta_0)\,\varrho_I \zeta} \Lambda \frac{L_t^P}{N_t}.$$

Generically, (51) has two solutions - $\overline{u}_L$ and $\overline{u}_H$. These are depicted in Figure (9) below. We will now show that $\overline{u}_H$ is in fact not a solution. To see this define the function

$$f(\overline{u}) = \overline{h} + (\zeta - 1)\,\varrho_I C^{\frac{\zeta}{\zeta-1}} \left(\Lambda \frac{L_t^P}{N_t}\right)^{-1} \overline{u}^{\frac{\zeta}{\zeta-1}}.$$

Note that

$$f'(\overline{u}_H) = \zeta \varrho_I \Omega^{\frac{\zeta}{\zeta-1}} \left(\Lambda \frac{L_t^P}{N_t}\right)^{-1} \overline{u}^{\frac{1}{\zeta-1}}.$$

Also note that $f'(\overline{u}_H) > 1$ as the curve intersects from below. This implies that

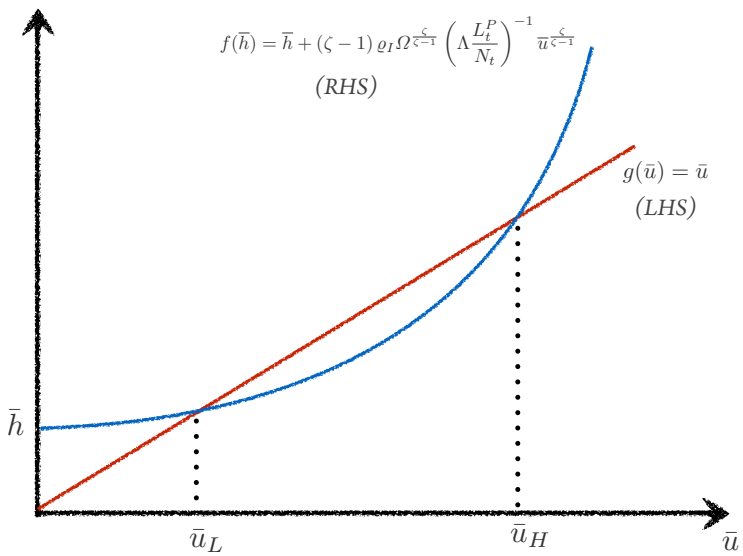
$$\zeta \varrho_I \Omega^{\frac{\zeta}{\zeta-1}} \left(\Lambda \frac{L_t^P}{N_t}\right)^{-1} \overline{u}^{\frac{1}{\zeta-1}} > 1.$$

Using that $I = (\Omega\overline{u})^{\frac{1}{\zeta-1}}$, this inequality implies that

$$\begin{aligned} 1 &< \zeta \varrho_I \Omega \left(\Lambda \frac{L_t^P}{N_t}\right)^{-1} (\Omega\overline{u})^{\frac{1}{\zeta-1}} \\ &= \frac{(\lambda^{\sigma-1} - 1)}{(g(\sigma-1) + \rho + \tau + \delta_0)} I. \end{aligned} \tag{52}$$

We are now going to show this solution for $I$ yields an unbounded solution for the value function.

Figure 9: Two Potential Solutions to Equation (51)




Notes: The figure displays the the left hand side (LHS) and right hand side (RHS) of equation (51).

Consider a firm with quality $q$ and $\Delta \geq \overline{\Delta}$. Substituting the guess (34) into the HJB equation yields

$$\begin{aligned}\overline{u}_H \frac{q^{\sigma-1} Y_t}{E[\mu^{1-\sigma}] N_t Q_t^{\sigma-1}} =& \overline{h} q^{\sigma-1} \frac{1}{E\left[\mu^{1-\sigma}\right]} \frac{Y_t}{Q_t^{\sigma-1}} \frac{1}{N_t} \\ &+ \max_I \left\{ I \frac{\overline{u}_H}{g(\sigma-1)+\rho+\tau+\delta_0} \frac{\left(\lambda^{\sigma-1}-1\right) q^{\sigma-1} Y_t}{E[\mu^{1-\sigma}] N_t Q_t^{\sigma-1}} - \frac{q^{\sigma-1}}{Q_t^{\sigma-1}} \varrho_I I^{\zeta} w_t \right\}.\end{aligned}$$

Using that $w = \frac{E\left[\mu_i^{-\sigma}\right]}{E\left[\mu_i^{1-\sigma}\right]} \frac{Y}{L^P}$, this equation yields

$$\overline{u}_H = \overline{h} + I^* \frac{\overline{u}_H \left(\lambda^{\sigma-1}-1\right)}{g(\sigma-1)+\rho+\tau+\delta_0} - \varrho_I \left(I^*\right)^{\zeta} \frac{E\left[\mu_i^{-\sigma}\right] N_t}{L^P},$$

where $I^*$ is the optimal innovation rate. Rearranging terms

$$\overline{u}_H = \frac{\overline{h} - \varrho_I \left(I^*\right)^{\zeta} \frac{E\left[\mu_i^{-\sigma}\right] N_t}{L^P}}{1 - \frac{I^*\left(\lambda^{\sigma-1}-1\right)}{g(\sigma-1)+\rho+\tau+\delta_0}}.$$

Note that $\overline{h} - \varrho_I \left(I^*\right)^{\zeta} \frac{E\left[\mu_i^{-\sigma}\right] N_t}{L^P}$ is simply the fundamental, per period cash flow of the firm. Hence, for

$\overline{u}_H$ to be finite, it has to be the case that

$$1 > \frac{I^* \left(\lambda^{\sigma-1} - 1\right)}{g(\sigma - 1) + \rho + \tau + \delta_0}.$$

This contradicts (52).